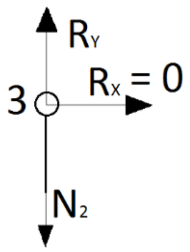
HOMEWORK 1

Assignment 1:

a) Explain from physics why the 5th row and column contain only zeros.

The node 3 only belongs to the bar 2, which is vertical. Therefore, it won´t have horizontal component (= x direction) because the bar 2 is only subjected to axial forces (=y direction). The forces acting upon the node 3 are the reaction forces (R_y, R_x) and the internal force (N_2).



The node 3 is only involved in the stiffness equations of the element 2 and, as it can be seen in the following equation system, the values are zero:

$$\begin{Bmatrix} f_{x1}^{(2)} \\ f_{x1}^{(2)} \\ f_{x3}^{(2)} \\ f_{x3}^{(2)} \end{Bmatrix} = \frac{EA^{(2)}}{L^{(2)}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1}^{(2)} \\ u_{x1}^{(2)} \\ u_{x3}^{(2)} \\ u_{x3}^{(2)} \end{Bmatrix}$$

b) Apply the BCs and show the modified stiffness system.

Every node can move vertically or horizontally, so if there are four nodes, there will be eight displacements possibilities. But some of them are known because of the boundary conditions. The nodes 2, 3 and 4 cannot move because they are fixed to the floor. However, the node 1 can move freely.

$$\frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & Symm & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ & & & & & & & c^3 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} H \\ -P \\ R_{x2} \\ R_{y2} \\ R_{x3} \\ R_{y3} \\ R_{x4} \\ R_{y4} \end{Bmatrix}$$

$$\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \end{Bmatrix} = \begin{Bmatrix} H \\ -P \end{Bmatrix}$$

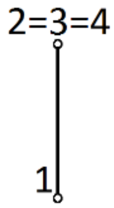
c) Solve for the displacements u_{x1} and u_{y1}. Check that the solution makes physical sense for the limit cases α→0 and α→π/2. Why does u_{x1} "blow up" if H ≠ 0 and α → 0?

The displacements are obtained solving the system of equation:

$$\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \end{Bmatrix} = \begin{Bmatrix} H \\ -P \end{Bmatrix} \rightarrow \begin{Bmatrix} u_{x1} \\ u_{y1} \end{Bmatrix} = \begin{Bmatrix} \dfrac{H}{2cs^2} \\ \dfrac{-P}{1+2c^3} \end{Bmatrix}$$

- **<u>α → 0:</u>**

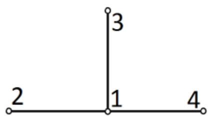
When the angle α is close to zero, the three bars will be located in the same place. The bars can only be subjected to axial forces. The vertical force (P) is divided equally into the three bars because their characteristics are the same ones. However, the horizontal ones (H) is a shear force and the bars do not support this type of forces. The following equations show numerically the explained concepts:



$$\begin{Bmatrix} u_{x1} \\ u_{y1} \end{Bmatrix} = \begin{Bmatrix} \dfrac{H}{2 \cdot 1 \cdot 0^2} \\ \dfrac{-P}{1+2 \cdot 1^3} \end{Bmatrix} = \begin{Bmatrix} \infty \\ -P/3 \end{Bmatrix} \rightarrow \begin{matrix} H = 2 \cdot 1 \cdot 0^2 u_{x1} = 0 \\ P = -(1+2 \cdot 1^3) u_{y1} = -3u_{y1} \end{matrix}$$

- **<u>α → π/2:</u>**

In this case, the force P is resisted by only one bar (bar 2), so the resistant area will be one third of the area of the previous case and consequently, the vertical displacement will be three times greater. Nevertheless, the BC of node 3 and 4 do not allow any horizontal displacement.



$$\begin{Bmatrix} u_{x1} \\ u_{y1} \end{Bmatrix} = \begin{Bmatrix} \dfrac{H}{2 \cdot 0 \cdot 1^2} \\ \dfrac{-P}{1+2 \cdot 0^3} \end{Bmatrix} \rightarrow \begin{matrix} H = 2 \cdot 0 \cdot 1^2 u_{x1} = 0 \\ P = -(1+2 \cdot 0^3) u_{y1} = -u_{y1} \end{matrix}$$

d) Recover the axial forces in the three members. Why do $F^{(1)}$ and $F^{(3)}$ "blow up" if H ≠ 0 and α → 0?

- **<u>Bar (1):</u>**

$$\begin{Bmatrix} \bar{u}_{x1} \\ \bar{u}_{y1} \\ \bar{u}_{x2} \\ \bar{u}_{y2} \end{Bmatrix} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{Bmatrix} \rightarrow \begin{Bmatrix} \bar{u}_{x1} = \dfrac{H}{2cs} + \dfrac{-Pc}{1+2c^3} \\ \bar{u}_{y1} = \dfrac{-H}{2s^2} + \dfrac{-sP}{1+2c^3} \\ \bar{u}_{x2} = 0 \\ \bar{u}_{y2} = 0 \end{Bmatrix}$$

$$F^{(1)} = \frac{AE}{L^{(1)}} (\bar{u}_{x2} - \bar{u}_{x1}) = \frac{AE}{L/c} \left[0 - \left(\frac{H}{2cs} + \frac{-Pc}{1+2c^3} \right) \right] = \frac{AE}{L} \left(\frac{-H}{2s} + \frac{Pc^2}{1+2c^3} \right)$$

- **Bar (2):**

$$\begin{Bmatrix}\bar{u}_{x1}\\ \bar{u}_{y1}\\ \bar{u}_{x3}\\ \bar{u}_{y3}\end{Bmatrix}=\begin{bmatrix}\mathrm{c} & s & 0 & 0\\ -s & c & 0 & 0\\ 0 & 0 & c & s\\ 0 & 0 & -s & c\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{y1}\\ u_{x3}\\ u_{y3}\end{Bmatrix}\rightarrow\begin{Bmatrix}\bar{u}_{x1}=\dfrac{-P}{1+2c^3}\\ \bar{u}_{y1}=\dfrac{-H}{2cs^2}\\ \bar{u}_{x3}=0\\ \bar{u}_{y3}=0\end{Bmatrix}$$

$$\mathrm{F}^{(3)}=\frac{AE}{L^{(3)}}\left(\bar{u}_{x3}-\bar{u}_{x1}\right)=\frac{AE}{L}\left[0-\left(\frac{-P}{1+2c^3}\right)\right]=\frac{AE}{L}\left(\frac{P}{1+2c^3}\right)$$

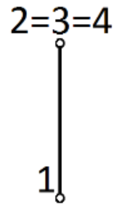
- **Bar (3):**

$$\begin{Bmatrix}\bar{u}_{x1}\\ \bar{u}_{y1}\\ \bar{u}_{x4}\\ \bar{u}_{y4}\end{Bmatrix}=\begin{bmatrix}\mathrm{s} & c & 0 & 0\\ -c & s & 0 & 0\\ 0 & 0 & s & c\\ 0 & 0 & -c & s\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{y1}\\ u_{x4}\\ u_{y4}\end{Bmatrix}\rightarrow\begin{Bmatrix}\bar{u}_{x1}=\dfrac{H}{2cs}+\dfrac{-Pc}{1+2c^3}\\ \bar{u}_{y1}=\dfrac{-H}{2s^2}+\dfrac{-sP}{1+2c^3}\\ \bar{u}_{x4}=0\\ \bar{u}_{y4}=0\end{Bmatrix}$$

$$\mathrm{F}^{(3)}=\frac{AE}{L^{(3)}}\left(\bar{u}_{x4}-\bar{u}_{x1}\right)=\frac{AE}{L/c}\left[0-\left(\frac{H}{2cs}+\frac{-Pc}{1+2c^3}\right)\right]=\frac{AE}{L}\left(\frac{-H}{2s}+\frac{Pc^2}{1+2c^3}\right)$$

Following with the reasoning given in the previous section (**α → 0**), if the three bars are located vertically, they cannot support horizontal forces. Therefore, the internal forces are in equilibrium only with the vertical one (P).

$$\mathrm{F}^{(3)}=\frac{AE}{L}\left(\frac{-H}{2s}+\frac{Pc^2}{1+2c^3}\right)=\frac{AE}{L}\left(\frac{-H}{2\cdot 0}+\frac{Pc^2}{1+2\cdot 1^3}\right)=\frac{AE}{L}\frac{Pc^2}{3}$$



Assignment 2:

The response of the structure due to the same forces will be the same for any distribution of nodes. If the inputs remain the same, logically the displacements in the nodes must continue being the same ones. This assumption is demonstrated numerically in the following equations.

The globalized stiffness equation for each element is defined as:

$$\begin{Bmatrix} f_{xi}^{(e)} \\ f_{yi}^{(e)} \\ f_{xj}^{(e)} \\ f_{yj}^{(e)} \end{Bmatrix} = \frac{AE^{(e)}}{L} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix} \begin{Bmatrix} u_{xi}^{(e)} \\ u_{yi}^{(e)} \\ u_{xj}^{(e)} \\ u_{yj}^{(e)} \end{Bmatrix}$$

- Bar 1 (1→2):

$$\begin{Bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{Bmatrix} = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{x2} \\ 0 \end{Bmatrix}$$

- Bar 2 (2→3):

$$\begin{Bmatrix} f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{Bmatrix} = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{x2} \\ 0 \\ u_{x3} \\ u_{y3} \end{Bmatrix}$$

- Bar 3 (1→4):

$$\begin{Bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x4}^{(3)} \\ f_{y4}^{(3)} \end{Bmatrix} = 40 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{x4} \\ u_{y4} \end{Bmatrix}$$

- Bar 4 (4→3):

$$\begin{Bmatrix} f_{x4}^{(4)} \\ f_{y4}^{(4)} \\ f_{x3}^{(4)} \\ f_{y3}^{(4)} \end{Bmatrix} = 40 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} u_{x4} \\ u_{y4} \\ u_{x3} \\ u_{y3} \end{Bmatrix}$$

The global stiffness equation is obtained assembling the previous matrixes and the determinant will continue being zero (singular):

$$\frac{EA}{L}\begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ & & 10 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 5 & 0 & -5 & 0 & 0 \\ & & & & 20 & 20 & -20 & -20 \\ & Symm & & & 20 & 25 & -20 & -20 \\ & & & & & & 40 & 40 \\ & & & & & & 40 & 40 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{x2} \\ 0 \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{Bmatrix} = \begin{Bmatrix} R_{x1} \\ R_{y1} \\ 0 \\ R_{y2} \\ 2 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

Finally, the linear system of equation would provide the values of the unknown variables. However, it is not necessary to solve the complete equation system to calculate the unknown displacements. The system could be simplified by obviating the arrays and columns corresponding to the restricted movements (u =0) in order to calculate only the unknown displacements:

$$\begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 20 & 20 & -20 & -20 \\ 0 & 20 & 25 & -20 & -20 \\ 0 & -20 & -20 & 40 & 40 \\ 0 & -20 & -20 & 40 & 40 \end{bmatrix}\begin{Bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.4 \\ -0.2 \\ 0.05 \\ 0.05 \end{Bmatrix}$$

As it has been assumed, the solutions are the same as the ones obtained in the original example. Putting an extra node only provides the displacements in this particular point, but the values of the displacement in the other points are the same ones.