Computational Structural Mechanics and Dynamics
Assignment 10

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08/05/2016

1 Solid and Structural Dynamics:

1. In the dynamic system of slide 6, let $r(t)$ be a constant force F. What is the effect of F on the time-dependent displacement $u(t)$ and the natural frequency of vibration of the system?

We know that this is a problem of forced undamped vibration system. If constant external force $\mathbf{F}$=$r(t)$ is applied on the system, the displacements will be oscillating/sinusoidal as we do not have any damper attached to the system. But actually in real systems, energy will be dissipated, i.e system will be damped, but often damping will be very small. And due to applied constant force **F**, if the system is being forced to vibrate at its natural frequency, resonance will occur and we will observe a large amplitude vibrations.

2. A weight whose mass is m is placed at the middle of a uniform axial bar of length L that is clamped at both ends. The mass of the bar may be neglected. Estimate the natural frequency of vibration in terms of m, L, E and A.

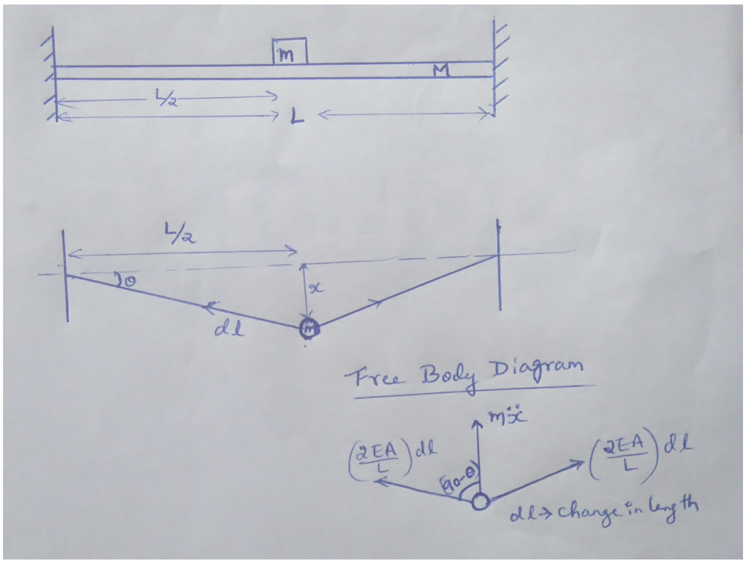

Figure 1: Axial bar before and after deformation upon mass and FBD

From the above figure we can see that axial bar of mass (M) (before and after deformation) of length L which is clamped at both ends and mass(m) is placed at the middle. Also free body diagram (FBD)is also shown for calculation purposes. We know that from Newtons Law,

$$F = ma$$

Therefore using this, from freebody diagram we obtain

$$(dl)(\frac{2EA}{L}) \times 2 \times sin\theta cos\theta + m\ddot{x} = 0$$

$$(dl)(\frac{4EA}{L}) \times sin\theta cos\theta + m\ddot{x} = 0$$

$$let \quad \theta \simeq small \quad hence \quad cos\theta = 1 \quad and \quad dl = \frac{x}{sin\theta}$$

Therefore upon substituting we get,

$$\frac{4EA}{L}x + m\ddot{x} = 0 \tag{1}$$

$$kx + m\ddot{x} = 0 \tag{2}$$

Comparing equation (1) and (2), we get stiffnes, $k = \frac{4EA}{L}$

And we know that natural frequency of vibration is,

$$\omega_n = \sqrt{\frac{k}{m}}$$

Neglecting mass of bar we obtain,

$$\boxed{\omega_n = \sqrt{\frac{4EA}{mL}}}$$

3. Use the expression on slide 18 to derive the mass matrix of slide 17.

It was asked actually to use the expression $\mathbf{m} = \int \mathbf{N}^T\mathbf{N}\rho dV$ to obtain consistent element mass matrix,

$$\mathbf{m} = \begin{bmatrix} \frac{\rho AL}{3} & \frac{\rho AL}{6} \\ \frac{\rho AL}{6} & \frac{\rho AL}{3} \end{bmatrix}$$

For the 2-node prismatic bar element moving along x, the stiffness shape functions are,

$$N_i^e = 1 - (x - x_i)/L = 1 - \xi, \quad N_j^e = (x - x_i)/L = \xi$$

$$J = \frac{dx}{d\xi} = L$$

$$\mathbf{m} = \int_0^L \mathbf{N}^T\mathbf{N}\rho A dx \tag{3}$$

After substituting shape functions in equation(3),

$$\mathbf{m} = \rho A \int_0^1 [(1-\xi)(\xi)] \begin{bmatrix} (1-\xi) \\ (\xi) \end{bmatrix} J d\xi$$

$$\mathbf{m} = \rho AL \int_0^1 \begin{bmatrix} (1-\xi)^2 & (\xi-\xi^2) \\ (\xi-\xi^2) & \xi^2 \end{bmatrix} d\xi$$

Therefore upon integrating and applying the limits, we finally obtain consistent element mass matrix as,

$$\mathbf{m} = \begin{bmatrix} \frac{\rho AL}{3} & \frac{\rho AL}{6} \\ \frac{\rho AL}{6} & \frac{\rho AL}{3} \end{bmatrix}$$

4. Obtain also the mass matrix of a two-node, linear displacement element with a variable cross-sectional area that varies from A_1 to A_2.

We know that the consistent mass for a 2-node bar element of length L and constant mass density ρ is,

$$\mathbf{m} = \int_0^L \mathbf{N}^T \mathbf{N} \rho A dx \tag{4}$$

Let the varying cross-sectional area be ,

$$A = A_1(\xi) + A_2(1-\xi)$$

For the 2-node prismatic bar element moving along x, the stiffness shape functions are,

$$N_i^e = 1 - (x - x_i)/L = 1 - \xi, \quad N_j^e = (x - x_i)/L = \xi$$

$$J = \frac{dx}{d\xi} = L$$

Therefore upon substituting shape functions and A in equation (4),

$$\mathbf{m} = \rho L \int_0^1 \begin{bmatrix} A_1\xi(1-\xi)^2 + A_2(1-\xi)^3 & A_1(\xi-\xi^2)\xi + A_2(\xi-\xi^2)(1-\xi) \\ A_1(\xi-\xi^2)\xi + A_2(\xi-\xi^2)(1-\xi) & A_1\xi^3 + A_2\xi^2(1-\xi) \end{bmatrix} d\xi$$

After integrating and applying limits finally we obtain ,

$$\mathbf{m} = \rho L \begin{bmatrix} \frac{1}{12}A_1 + \frac{1}{4}A_2 & \frac{1}{12}A_1 + \frac{1}{12}A_2 \\ \frac{1}{12}A_1 + \frac{1}{12}A_2 & \frac{1}{4}A_1 + \frac{1}{12}A_2 \end{bmatrix}$$

Note: If we put A_1=A_2(for same cross-section), then we obtain the same mass matrix as previous question.

5. A uniform two-node bar element is allowed to move in a 3D space. The nodes have only translational d.o.f. What is the diagonal mass matrix of the element?

If a uniform two-node bar element length (L), cross sectional area(A) and constant mass density (ρ) is allowed to move in a 3D space and nodes have only translational d.o.f. but all rotational d.o.f are zero. Then the diagonal mass matrix of the element is,

$$\mathbf{m} = \begin{bmatrix} \frac{\rho AL}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\rho AL}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\rho AL}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\rho AL}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\rho AL}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\rho AL}{2} \end{bmatrix}$$