Assignment 3

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Assignment 3.1

a)

To go from the plane stress to the plane strain problem, the following equation must be verified:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E*}{1-\nu*^2}\begin{bmatrix} 1 & \nu* & 0 \\ \nu* & 1 & 0 \\ 0 & 0 & \frac{1-\nu*}{2} \end{bmatrix} = \frac{E\,(1-\nu)}{(1+\nu)\,(1-2\nu)}\begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}$$

$\nu*$ will be:

$$\boldsymbol{\nu* = \frac{\nu}{1-\nu}}$$

So:

$$\frac{E}{1-\nu*^2}\begin{bmatrix} 1 & \nu* & 0 \\ \nu* & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu*}{2} \end{bmatrix} = \frac{E*}{1-\frac{\nu^2}{(1-\nu)^2}}\begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & 0 \\ \dfrac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2\,(1-\nu)} \end{bmatrix}$$

Thus:

$$\frac{E*}{1-\dfrac{\nu^2}{(1-\nu)^2}} = \frac{E\,(1-\nu)}{(1+\nu)\,(1-2\nu)}$$

And:

$$\boldsymbol{E* = \frac{E}{(1+\nu)\,(1-\nu)}}$$

To go from the plane strain to the plane stress problem, the following equation must be verified:

$$\frac{E\,(1-\nu*)}{(1+\nu*)\,(1-2\nu*)}\begin{bmatrix} 1 & \dfrac{\nu*}{1-\nu*} & 0 \\ \dfrac{\nu*}{1-\nu*} & 1 & 0 \\ 0 & 0 & \dfrac{1-2\nu*}{2\,(1-\nu*)} \end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

$\nu*$ can be obtained from:

$$\frac{1-2\nu*}{2\,(1-\nu*)} = \frac{1-\nu}{2} \rightarrow \boldsymbol{\nu* = \frac{\nu}{1+\nu}}$$

Thus:

$$\frac{E\,(1-\nu*)}{(1+\nu*)\,(1-2\nu*)} = \frac{E}{1-\nu^2}$$

And:

$$\boldsymbol{E* = \frac{E\,(1+2\nu)}{(1+\nu)^2}}$$

b)

The relation between σ and e is:

$$\sigma = Ee$$

So:

$$U = \frac{1}{2}e^T Ee = \frac{1}{2}e^T\sigma = \frac{1}{2}\begin{bmatrix} e_x & e_y & 2e_{xy} \end{bmatrix}\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \frac{1}{2}\left(\sigma_x e_x + \sigma_y e_y + 2\sigma_{xy}e_{xy}\right)$$

Using $C = E^{-1}$:

$$e = C\sigma$$

Thus:

$$U = \frac{1}{2}\sigma^T C\sigma = \frac{1}{2}\sigma^T e = \frac{1}{2}\begin{bmatrix} \sigma_x & \sigma_y & \sigma_{xy} \end{bmatrix}\begin{bmatrix} e_x \\ e_y \\ 2e_{xy} \end{bmatrix} = \frac{1}{2}\left(\sigma_x e_x + \sigma_y e_y + 2\sigma_{xy}e_{xy}\right)$$

Assignment 3.2

1)

First we must compute the area of the element:

$$A = \frac{1}{2}\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 1/2\begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} = 2$$

Now we can compute the triangular coordinates of the triangle:

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{2A}\begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & -1 & -1 \\ 0 & 2 & -2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

The shape functions derivatives are:

$$B = \frac{1}{2A}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix}$$

The elemental stiffness matrix is:

$$K^e = B^T EB\int_{\Omega^e} hd\Omega = \frac{h}{4A}\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 2 & 0 & -2 \\ 0 & -2 & -2 \\ -1 & 0 & 3 \\ 0 & 3 & -1 \end{bmatrix}\begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & -2 \\ 0 & 0 & 50 \end{bmatrix}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix}$$

$$K^e = \begin{bmatrix} 18.75 & 9.375 & -12.5 & -6.25 & -6.25 & -3.125 \\ 9.375 & 18.75 & 6.25 & 12.5 & -15.625 & -31.25 \\ -12.5 & 6.25 & 75.0 & -37.5 & -62.5 & 31.25 \\ -6.25 & 12.5 & -37.5 & 75.0 & 43.75 & -87.5 \\ -6.25 & -15.625 & -62.5 & 43.75 & 68.75 & -28.125 \\ -3.125 & -31.25 & 31.25 & -87.5 & -28.125 & 118.75 \end{bmatrix}$$

2)

The sum of rows 1,3 and 5 and rows 2,4 and 6 must vanish:

$$\begin{bmatrix} 18.75 & 9.375 & -12.5 & -6.25 & -6.25 & -3.125 \end{bmatrix} + \begin{bmatrix} -12.5 & 6.25 & 75.0 & -37.5 & -62.5 & 31.25 \end{bmatrix} +$$
$$+ \begin{bmatrix} -6.25 & -15.625 & -62.5 & 43.75 & 68.75 & -28.125 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 9.375 & 18.75 & 6.25 & 12.5 & -15.625 & -31.25 \end{bmatrix} + \begin{bmatrix} -6.25 & 12.5 & -37.5 & 75.0 & 43.75 & -87.5 \end{bmatrix} +$$
$$+ \begin{bmatrix} -3.125 & -31.25 & 31.25 & -87.5 & -28.125 & 118.75 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The same happens for the columns:

$$\begin{bmatrix} 18.75 \\ 9.375 \\ -12.5 \\ -6.25 \\ -6.25 \\ -3.125 \end{bmatrix} + \begin{bmatrix} -12.5 \\ 6.25 \\ 75.0 \\ -37.5 \\ -62.5 \\ 31.25 \end{bmatrix} + \begin{bmatrix} -6.25 \\ -15.625 \\ -62.5 \\ 43.75 \\ 68.75 \\ -28.125 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 9.375 \\ 18.75 \\ 6.25 \\ 12.5 \\ -15.625 \\ -31.25 \end{bmatrix} + \begin{bmatrix} -6.25 \\ 12.5 \\ -37.5 \\ 75.0 \\ 43.75 \\ -87.5 \end{bmatrix} + \begin{bmatrix} -3.125 \\ -31.25 \\ 31.25 \\ -87.5 \\ -28.125 \\ 118.75 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

All this is due to the fact that the elemental stiffness matrix is singular and the element needs a constraint to be able to hold an external load. If no constrains are imposed, then there will be infinite different possible displacements for a given set of external forces. The sum of the odd (or even) rows (or columns) is 0 is because rigid body motions are allowed when no constrain is applied, such that:

$$\text{For } \mathbf{u} = \begin{bmatrix} a \\ b \\ a \\ b \\ a \\ b \end{bmatrix}, \mathbf{Ku} = \mathbf{0} \; \forall a, b \in \mathbb{R}$$