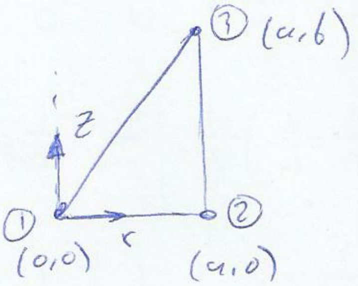
A. 5. 1

1)



$$E = E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ r \\ z \\ u_r \\ u_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \\ u_{r1} & u_{r2} & u_{r3} \\ u_{z1} & u_{z2} & u_{z3} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \longrightarrow \begin{aligned} 1 &= N_1 + N_2 + N_3 \\ r &= r_1 N_1 + r_2 N_2 + r_3 N_3 \\ z &= z_1 N_1 + z_2 N_2 + z_3 N_3 \end{aligned}$$

For $z_1 = z_2 = r_1 = 0$, $r_2 = r_3 = a$, $z_3 = b$

$$\left.\begin{aligned} 1 &= N_1 + N_2 + N_3 \\ r &= a(N_2 + N_3) \\ z &= b N_3 \end{aligned}\right\} \quad N_1 = 1 - \frac{r}{a}, \quad N_2 = \frac{r}{a} - \frac{z}{b}, \quad N_3 = \frac{z}{b}$$

$$q_r = \begin{bmatrix} \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial r} \end{bmatrix} = \begin{bmatrix} -\frac{1}{a} & \frac{1}{a} & 0 \end{bmatrix}$$

$$q_z = \begin{bmatrix} \frac{\partial N}{\partial z} & \frac{\partial N_2}{\partial z} & \frac{\partial N_3}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{b} & \frac{1}{b} \end{bmatrix}$$

$$q_\phi = \begin{bmatrix} \frac{N_1}{r} & \frac{N_2}{r} & \frac{N_3}{r} \end{bmatrix} = \begin{bmatrix} \frac{1}{r} - \frac{1}{a} & \frac{1}{a} - \frac{z}{br} & \frac{z}{br} \end{bmatrix}$$

$$B = \begin{bmatrix} q_r & 0 \\ 0 & q_z \\ q_\phi & 0 \\ q_z & q_r \end{bmatrix} = \begin{bmatrix} -\frac{1}{a} & \frac{1}{a} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{b} & -\frac{1}{b} \\ \frac{1}{r} - \frac{1}{a} & \frac{1}{a} - \frac{z}{br} & \frac{z}{br} & 0 & 0 & 0 \\ 0 & -\frac{1}{b} & \frac{1}{b} & -\frac{1}{a} & \frac{1}{a} & 0 \end{bmatrix}$$

$$
B^T E B = E \begin{bmatrix}
1/a^2 + \left(\frac{1}{r} - \frac{1}{a}\right)^2 & -\frac{1}{a^2} + \left(\frac{1}{r} - \frac{1}{a}\right)\left(\frac{1}{a} - \frac{z}{br}\right) & \frac{z}{br}\left(\frac{1}{r} - \frac{1}{a}\right) & 0 & 0 & 0 \\
 & \frac{1}{a^2} + \left(\frac{1}{a} - \frac{z}{br}\right)^2 + \frac{1}{2b^2} & \frac{z}{br}\left(\frac{1}{a} - \frac{z}{br}\right) - \frac{1}{2b^2} & \frac{1}{2ab} & \frac{-1}{2ab} & 0 \\
 & & \frac{z^2}{b^2 r^2} + \frac{1}{2b^2} & -\frac{1}{2ab} & +\frac{1}{2ab} & 0 \\
 & \text{Sym} & & \frac{1}{2a^2} & -\frac{1}{2a^2} & 0 \\
 & & & & \frac{1}{b^2} + \frac{1}{2a^2} & -\frac{1}{b^2} \\
 & & & & & 1/b^2
\end{bmatrix}
$$

where:

$$K^e = 2\pi \int_0^b \int_0^a B^T E B \, r \, dr \, dz$$

To simplify terms:

$$B^T E B = M^e$$

$$M^e_{11} = \frac{1}{a^2} + \frac{1}{r^2} - \frac{2}{ar} + \frac{1}{a^2}$$

$$M^e_{12} = -\frac{1}{a^2} + \frac{1}{ar} - \frac{1}{a^2} - \frac{z}{bra}$$

$$M^e_{13} = \frac{z}{br^2} - \frac{z}{bra}$$

$$M^e_{22} = \frac{1}{a^2} + \frac{1}{a^2} - \frac{2z}{bra} + \frac{z^2}{b^2 r^2} + \frac{1}{2b^2}$$

$$M^e_{23} = \frac{z}{bra} - \frac{z^2}{b^2 r^2} - \frac{1}{2b^2}$$

$$M^e_{33} = \frac{z^2}{b^2 r^2} + \frac{1}{2b^2}$$

$$M^e_{55} = \frac{1}{b^2} + \frac{1}{2a^2}$$

then terms are integrated:

$$\int_0^b \int_0^a \frac{1}{a^2} r \, dr \, dz = \frac{b}{2} \qquad \int_0^b \int_0^a \frac{1}{r^2} r \, dr \, dz = b \ln a$$

$$\int_0^b \int_0^a \frac{z}{ra} r \, dr \, dz = 2b \qquad \int_0^b \int_0^a \frac{1}{ra} r \, dr \, dz = b$$

$$\int_0^b \int_0^a \frac{z}{bra} r \, dr \, dz = \frac{b}{2} \qquad \int_0^b \int_0^a \frac{1}{2b^2} r \, dr \, dz = \frac{a^2}{b}$$

$$\int_0^b \int_0^a \frac{z^2}{b^2 r^2} r \, dr \, dz = b \ln a + 2b \qquad \int_0^b \int_0^a \frac{z}{b r^2} r \, dr \, dz = b \ln a + b$$

$$\int_0^b \int_0^a \frac{1}{2ab} r \, dr \, dz = \frac{a}{4}$$

~~The different components of K^e are:~~

$\cancel{K^e_{11} =}$ $K^e_{11} =$

The different term of M^e after integration are:

$$M^e_{11} = b \ln a - b$$

$$M^e_{12} = -\frac{b}{2}$$

$$M^e_{13} = b \ln a - \frac{b}{2}$$

$$M^e_{22} = 2b + b \ln a + \frac{a^2}{b}$$

$$M^e_{23} = -\frac{3}{2}b - b \ln a - \frac{a^2}{b}$$

$$M^e_{33} = b \ln a + 2b + \frac{a^2}{b}$$

$$M^e_{55} = \frac{a^2}{2b} + \frac{b}{4}$$

where the stiffness matrix is:

$$K^e = 2\pi E \begin{bmatrix} b\ln a - b & -\frac{b}{2} & b\ln a - \frac{b}{2} & 0 & 0 & 0 \\ & \frac{5}{2}b + b\ln a + \frac{a^2}{b} & -\frac{3}{2}b - b\ln a - \frac{a^2}{b} & \frac{a}{4} & -\frac{a}{4} & 0 \\ & & b\ln a + 2b + \frac{a^2}{b} & -\frac{a}{4} & \frac{a}{4} & 0 \\ & \text{Sym} & & \frac{b}{4} & -\frac{b}{4} & 0 \\ & & & & \frac{a^2}{2b} + \frac{b}{4} & -\frac{a^2}{2b} \\ & & & & & \frac{a^2}{2b} \end{bmatrix}$$

2)

$$R1 = b\ln a - b - \frac{b}{2} + b\ln a - \frac{b}{2} = 2b(\ln a - 1) \neq 0$$

$$R2 = -\frac{b}{2} + 2b + b\ln a + \frac{a^2}{b} - \frac{3}{2}b - b\ln a - \frac{a^2}{b} + \frac{a}{4} - \frac{a}{4} = 0 \;//$$

$$R3 = b\ln a - \frac{b}{2} - \frac{3}{2}b - b\ln a - \frac{a^2}{b} + b\ln a + 2b + \frac{a^2}{b} - \frac{a^2}{4} + \frac{a^2}{4} \neq 0$$

$$R4 = \frac{a}{4} - \frac{a}{4} + \frac{b}{4} - \frac{b}{4} = 0 \;//$$

$$R5 = -\frac{a}{4} + \frac{a}{4} - \frac{b}{4} + \frac{a^2}{2b} + \frac{b}{4} - \frac{a^2}{2b} = 0$$

$$R6 = -\frac{1}{b^2} + \frac{1}{b^2} = 0 \;//$$

3) $$f_{ext} = \int_0^b \int_0^a N^T b\, r\, dr\, dz$$

$$N = \begin{bmatrix} 1-\frac{r}{a} & \frac{r}{a}-\frac{z}{b} & \frac{z}{b} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\frac{r}{a} & \frac{r}{a}-\frac{z}{b} & \frac{z}{b} \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

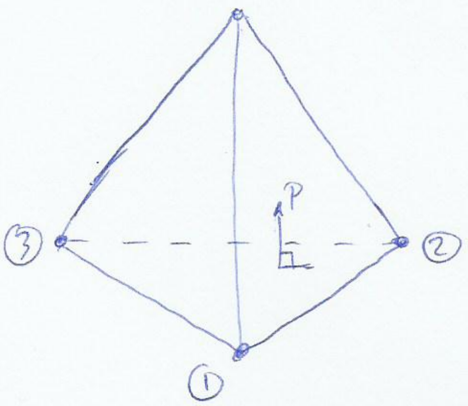
$$N^T b r = \begin{bmatrix} 0 & 0 & 0 & -gr+\frac{gr^2}{a} & -\frac{gr^2}{a}+\frac{gzr}{b} & -\frac{zgr}{b} \end{bmatrix}$$

$$\int_a^a N^T b r\, dr = \begin{bmatrix} 0 & 0 & 0 & -\frac{ga^2}{6} & -\frac{ga^2}{3}+\frac{gza^2}{2b} & -\frac{gza^2}{2b} \end{bmatrix} = [P]$$

$$\int_0^b [P]\, dz = \begin{bmatrix} 0 & 0 & 0 & -\frac{ga^2z}{6}\Big|_{z=b} & -\frac{ga^2z}{3}+\frac{gz^2a^2}{4b}\Big|_{z=b} & -\frac{gz^2a^2}{4b}\Big|_{z=b} \end{bmatrix}$$

$$\underline{\underline{f^e_{ext}}} = \begin{bmatrix} 0 & 0 & 0 & -\frac{ba^2g}{6} & -\frac{ba^2g}{12} & -\frac{ba^2g}{4} \end{bmatrix}$$

5.2



Area$_{123}$ of the base

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\left[(x_2 y_3 + x_1 y_2 + x_3 y_1) - (y_1 x_2 + y_2 x_3 + x_1 y_3)\right]$$

$$= \frac{1}{2}\left[(x_2 y_3 - y_2 x_3) - x_1(y_3 - y_2) + y_1(x_3 - x_2)\right]$$

Normal vector is calculated by vectorial product of $\overline{12}$ and $\overline{13}$

$$\bar{n} = \overline{12} \times \overline{13} \quad \begin{vmatrix} i & j & k \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$$

$$\bar{n} = \left[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)\right]\bar{i} + \left[(x_3 - x_1)(z_2 - z_1) - (x_2 - x_1)(z_3 - z_1)\right]\bar{j}$$
$$+ \left[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)\right]\bar{k}$$

Tensor verctor is $\bar{t} = p \cdot \bar{n}$

then $f^e = A \cdot \bar{t} = A p \bar{n}$

$$f^e = \frac{1}{2} p \left[(x_2 y_3 - y_2 x_3) - x_1(y_3 - y_2) + y_1(x_3 - x_2)\right] \begin{bmatrix} (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) \\ (x_3 - x_1)(z_2 - z_1) - (x_2 - x_1)(z_3 - z_1) \\ (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) \end{bmatrix}$$