Computational Structural Mechanics and Dynamics

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1 To show the master stiffness equations, we use the following formula to get the stiffness equation for each element,

$$\frac{E^eA^e}{L^e}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}\begin{bmatrix} u_{xi}^{(e)} \\ u_{yi}^{(e)} \\ u_{xj}^{(e)} \\ u_{yj}^{(e)} \end{bmatrix}=\begin{bmatrix} f_{xi}^{(e)} \\ f_{yi}^{(e)} \\ f_{xj}^{(e)} \\ f_{yj}^{(e)} \end{bmatrix}$$

The transformation matrix is

$$\mathrm{T}=\begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

By assuming the angle between the wall (along x-coordinate) and the truss is β, where $\beta+\alpha=\frac{\pi}{2}$, as shown in figure 1.

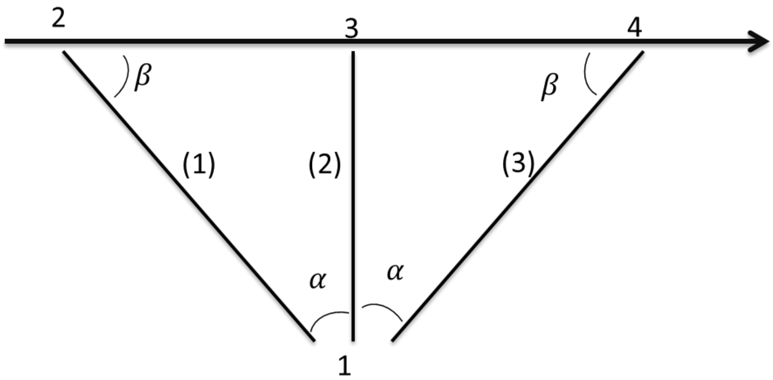

Figure 1 geometry analysis and notation explanation

For element 1, we have

$$\sin(-\beta)=-\mathrm{a}=\mathrm{c},\cos(-\beta)=\mathrm{b}=\mathrm{s}$$

thus,

$$\frac{E^{(1)}A^{(1)}}{L^{(1)}}\begin{bmatrix} b^2 & -ab & -b^2 & ab \\ -ab & a^2 & ab & -a^2 \\ -b^2 & ab & b^2 & -ab \\ ab & -a^2 & -ab & a^2 \end{bmatrix}\begin{bmatrix} u_{x2}^{(1)} \\ u_{y2}^{(1)} \\ u_{x1}^{(1)} \\ u_{y1}^{(1)} \end{bmatrix}=\begin{bmatrix} f_{x2}^{(1)} \\ f_{y2}^{(1)} \\ f_{x1}^{(1)} \\ f_{y1}^{(1)} \end{bmatrix}$$

For element 2,we have

$$\sin\left(-\frac{\pi}{2}\right)=-1,\cos(-\frac{\pi}{2})=0$$

thus,

$$\frac{E^{(2)}A^{(2)}}{L^{(2)}}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_{x3}^{(2)} \\ u_{y3}^{(2)} \\ u_{x1}^{(2)} \\ u_{y1}^{(2)} \end{bmatrix} = \begin{bmatrix} f_{x3}^{(2)} \\ f_{y3}^{(2)} \\ f_{x1}^{(2)} \\ f_{y1}^{(2)} \end{bmatrix}$$

For element 3, we have

$$\sin(\pi-\beta) = \sin(\beta) = \cos(\alpha) = c, \cos(\pi-\beta) = -\sin(\alpha) = -s$$

thus,

$$\frac{E^{(3)}A^{(3)}}{L^{(3)}}\begin{bmatrix} s^2 & -sc & -s^2 & sc \\ -sc & c^2 & sc & -c^2 \\ -s^2 & sc & s^2 & -sc \\ sc & -c^2 & -sc & c^2 \end{bmatrix}\begin{bmatrix} u_{x4}^{(3)} \\ u_{y4}^{(3)} \\ u_{x1}^{(3)} \\ u_{y1}^{(3)} \end{bmatrix} = \begin{bmatrix} f_{x4}^{(3)} \\ f_{y4}^{(3)} \\ f_{x1}^{(3)} \\ f_{y1}^{(3)} \end{bmatrix}$$

According to the known conditions, that,

$$L^{(3)} = L^{(1)} = \frac{L}{a} = \frac{L}{c}$$

$a = c, b = s$

The three stiffness equations are written in terms of s,c

Local stiffness equation element 1:

$$\frac{EA}{L}\begin{bmatrix} cs^2 & c^2s & -cs^2 & -c^2s \\ c^2s & c^3 & -c^2s & -c^3 \\ -cs^2 & -c^2s & cs^2 & c^2s \\ -c^2s & -c^3 & c^2s & c^3 \end{bmatrix}\begin{bmatrix} u_{x2}^{(1)} \\ u_{y2}^{(1)} \\ u_{x1}^{(1)} \\ u_{y1}^{(1)} \end{bmatrix} = \begin{bmatrix} f_{x2}^{(1)} \\ f_{y2}^{(1)} \\ f_{x1}^{(1)} \\ f_{y1}^{(1)} \end{bmatrix}$$

Local stiffness equation element 2:

$$\frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_{x3}^{(2)} \\ u_{y3}^{(2)} \\ u_{x1}^{(2)} \\ u_{y1}^{(2)} \end{bmatrix} = \begin{bmatrix} f_{x3}^{(2)} \\ f_{y3}^{(2)} \\ f_{x1}^{(2)} \\ f_{y1}^{(2)} \end{bmatrix}$$

Local stiffness equation for element 3:

$$\frac{EA}{L}\begin{bmatrix} cs^2 & -c^2s & -cs^2 & c^2s \\ -c^2s & c^3 & c^2s & -c^3 \\ -cs^2 & c^2s & cs^2 & -c^2s \\ c^2s & -c^3 & -c^2s & c^3 \end{bmatrix}\begin{bmatrix} u_{x4}^{(3)} \\ u_{y4}^{(3)} \\ u_{x1}^{(3)} \\ u_{y1}^{(3)} \end{bmatrix} = \begin{bmatrix} f_{x4}^{(3)} \\ f_{y4}^{(3)} \\ f_{x1}^{(3)} \\ f_{y1}^{(3)} \end{bmatrix}$$

Because each node has two degree of freedoms, the total stiffness matrix should be a matrix which has eight rows and eight columns. The next step is the assembly process with two rules. The first rule is compatibility which means that the joint displacements of all members meeting at a joint must be the same. Besides the equilibrium rule is that the sum of forces exerted by all members that meet at a joint must balance the external force applied to that joint. With the two rules, we get the global stiffness matrix equation.

Expanded element stiffness equation for element 1:

$$\frac{EA}{L}\begin{bmatrix} cs^2 & c^2s & 0 & 0 & 0 & 0 & -cs^2 & -c^2s \\ c^2s & c^3 & 0 & 0 & 0 & 0 & -c^2s & -c^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -cs^2 & -c^2s & 0 & 0 & 0 & 0 & cs^2 & c^2s \\ -c^2s & -c^3 & 0 & 0 & 0 & 0 & c^2s & c^3 \end{bmatrix}\begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \\ u_{3}^{(1)} \\ u_{3}^{(1)} \\ u_{4}^{(1)} \\ u_{4}^{(1)} \end{bmatrix} = \begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \\ f_{3}^{(1)} \\ f_{3}^{(1)} \\ f_{4}^{(1)} \\ f_{4}^{(1)} \end{bmatrix}$$

Expanded element stiffness equation for element 1:

$$\frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_{x1}^{(2)} \\ u_{y1}^{(2)} \\ u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{3}^{(2)} \\ u_{3}^{(2)} \\ u_{4}^{(2)} \\ u_{4}^{(2)} \end{bmatrix} = \begin{bmatrix} f_{x1}^{(2)} \\ f_{y1}^{(2)} \\ f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{3}^{(2)} \\ f_{3}^{(2)} \\ f_{4}^{(2)} \\ f_{4}^{(2)} \end{bmatrix}$$

Expanded element stiffness equation for element 1:

$$\frac{EA}{L}\begin{bmatrix} cs^2 & -c^2s & -cs^2 & c^2s & 0 & 0 & 0 & 0 \\ -c^2s & c^3 & c^2s & -c^3 & 0 & 0 & 0 & 0 \\ -c^2s & c^2s & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ c^2s & -c^3 & -c^2s & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_{x1}^{(3)} \\ u_{y1}^{(3)} \\ u_{x2}^{(3)} \\ u_{y2}^{(3)} \\ u_{3}^{(3)} \\ u_{3}^{(3)} \\ u_{4}^{(3)} \\ u_{4}^{(3)} \end{bmatrix} = \begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x2}^{(3)} \\ f_{y2}^{(3)} \\ f_{3}^{(3)} \\ f_{3}^{(3)} \\ f_{4}^{(3)} \\ f_{4}^{(3)} \end{bmatrix}$$

By the equilibrium rule, we obtain the master stiffness equation as follows:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ 0 & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ -cs^2 & c^2s & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ c^2s & -c^3 & -c^2s & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -cs^2 & -c^2s & 0 & 0 & 0 & 0 & cs^2 & c^2s \\ -c^2s & -c^3 & 0 & 0 & 0 & 0 & c^2s & c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, we have proved the given equation.

Reasons that the row and column is zero in 5th can be explained from two aspects, mathematically and physically. We know that the stiffness matrix is the description for the property of the subject. In this case, the 5th row and column describe the displacement for node 3 in x coordinate. Since the geometry is symmetric, the zero-element is the matrix means the displacement of node 3 in x direction would not have any influence on other nodes.

2 According to the figure 1.1, node 2, node 3 and node 4 are fixed point. We thus know that the boundary condition is that,

$$[u_{x2} \quad u_{y2} \quad u_{x3} \quad u_{y3} \quad u_{x4} \quad u_{y4}]^T = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T$$

So we can obtain the modified stiffness system:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}$$

This is 2-equation modified stiffness system!

3 Continually, we can solve for u_{x1} and u_{y1},

$$\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} \dfrac{HL}{2EAcs^2} \\ \dfrac{-PL}{EA(1+2c^3)} \end{bmatrix}$$

When $\alpha \to 0$,through Taylor expansion, we know that $s = \alpha$, $c = 1$.So $u_{x1} \to \infty$ and $u_{y1} \to \frac{-PL}{3EA}$ In this case, we can think that the system has only one truss whose cross section area is 3A, so in y direction , we have $u_{y1} \to \frac{-PL}{3EA}$. While in x direction, the solution doesn't make sense. A very small H would make u_{x1} very large, which seems that the truss system is blown up.

When $\alpha \to \frac{\pi}{2}$, again through Taylor expansion, we obtain $u_{x1} \to \infty$ and $u_{y1} \to \frac{-PL}{EA}$. In this case, we can think that the system has only one truss whose cross section area is A, so in y direction, we have $u_{y1} \to \frac{-PL}{EA}$. However, in x direction, when $\alpha \to \frac{\pi}{2}$, element 1 and element 2 is going to be parallel to the x direction, so there is no boundary condition for node 2 and node 3, the displacement for node 1 in x direction can be any larger infinite.

4 we firstly compute the local displacement through the transformation matrix.
For element 1

$$\begin{bmatrix} u_{x2}^{(loc1)} \\ u_{y2}^{(loc1)} \\ u_{x1}^{(loc1)} \\ u_{y1}^{(loc1)} \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dfrac{HL}{2EAcs^2} \\ \dfrac{-PL}{EA(1+2c^3)} \end{bmatrix}$$

So we have $u_{x1}^{(loc1)} = \frac{HL}{2EAs^2} - \frac{PLs}{EA(1+2c^3)}$, $u_{x2}^{(loc1)} = 0$

The elongation is d1= $u_{x1}^{(loc1)} - u_{x2}^{(loc1)} = \frac{HL}{2EAs^2} - \frac{PLs}{EA(1+2c^3)}$

So axial force $F^{(1)} = \frac{EAc}{L} * d1 = \frac{Hc}{2s^2} - \frac{Pcs}{1+2c^3}$

For element 2

$$\begin{bmatrix} u_{x3}^{(loc2)} \\ u_{y3}^{(loc2)} \\ u_{x1}^{(loc2)} \\ u_{y1}^{(loc2)} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{HL}{2EAcs^2} \\ \frac{-PL}{EA(1+2c^3)} \end{bmatrix}$$

So we have $u_{x1}^{(loc2)} = \frac{PL}{EA(1+2c^3)}$, $u_{x4}^{(loc2)} = 0$

The elongation is d2= $u_{x4}^{(loc2)} - u_{x1}^{(loc2)} = \frac{PL}{EA(1+2c^3)}$

So axial force $F^{(2)} = \frac{EA}{L} * d2 = \frac{P}{1+2c^3}$

For element 3

$$\begin{bmatrix} u_{x4}^{(loc3)} \\ u_{y4}^{(loc3)} \\ u_{x1}^{(loc3)} \\ u_{y1}^{(loc3)} \end{bmatrix} = \begin{bmatrix} -s & c & 0 & 0 \\ -c & -s & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & -c & -s \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{HL}{2EAcs^2} \\ \frac{-PL}{EA(1+2c^3)} \end{bmatrix}$$

So we have $u_{x1}^{(loc3)} = -\frac{HL}{2EAcs} - \frac{PLc}{EA(1+2c^3)}$, $u_{x4}^{(loc3)} = 0$

The elongation is d3= $u_{x4}^{(loc3)} - u_{x1}^{(loc3)} = -\frac{HL}{2EAcs} - \frac{PLc}{EA(1+2c^3)}$

So axial force $F^{(3)} = \frac{EAc}{L} * d3 = -\frac{H}{2s} - \frac{Pc^2}{(1+2c^3)}$ (Compression)

Now we can understand why $F^{(3)}$ and $F^{(1)}$ "blow up" if $\mathrm{H} \neq 0$ and $\alpha \to 0$. we find that in this case, $F^{(3)} \to \infty$ and $F^{(1)} \to \infty$. For the truss, we apply even a very small H would result in an infinite axial internal force. This is impossible for the truss to maintain equilibrium in the real case.

5 References

The Direct Stiffness Method, Course Slides, Lecture: Miguel Cervera