Universitat Politecnica De Catalunya

Computational Structural Mechanics
Dynamics

Dynamics
Assignment-9

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(1) In the dynamic system of slide 6, let r(t) be a constant force F. What is the effect of F on the time-dependent displacement u(t) and the natural frequency of vibration of the system?

Sol. 1

$$F = m\ddot{u} + ku \Rightarrow \frac{F}{k} = \frac{m\ddot{u}}{k} + u$$

Solution of this equation is as follows:

$$u = \frac{F}{K} + A\sin(\omega t) + B\cos(\omega t) = \frac{F}{K} + \bar{u}\sin(\omega t + \theta)$$

where $\bar{u}$ is the amplitude of vibration, ω is the natural frequency of vibration
As a constant force in applied on an undamped system, the system will be oscillating sinusoidally. Further, if the situation arises such that the frequency of oscillation becomes equal to the natural frequency due to the load applied, there will be resonance condition resulting in enlarged amplitudes. In the end, it can be deduced from the equation that there is no dependency of natural frequency of vibration on the F. Natural frequency, $\omega = \sqrt{\frac{k}{m}}$

(2) A weight whose mass is m is placed at the middle of a uniform axial bar of length L that is clamped at both ends. The mass of the bar may be neglected. Estimate the natural frequency of vibration in terms of m, L, E and A. Suggestion: First determine the effective k.

Sol. 2 The deflection at the center, i.e. at x=L/2, will be maximum given by :

$$\delta = \frac{Fl^3}{192EI}$$

$$F = k \Rightarrow \delta = \frac{F}{k}, therefore, k = \frac{192EI}{l^3}$$

We consider that the bar has square cross-section, thus, I=$\frac{L^4}{12}$. Natural frequency is given by:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{192EI}{mL^3}} = \sqrt{\frac{192EL^4}{12mL^3}} = \frac{4A}{L}\sqrt{\frac{E}{mL}}$$

(3) Use the expression on slide 18 to derive the mass matrix of slide 17.

Sol. 3

$$m = \int N^T N \rho dV, where N = \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix}$$

$$m = \frac{\rho A}{4} \int_{-1}^{+1} \begin{bmatrix} (1-\xi)^2 & (1-\xi^2) \\ (1-\xi^2) & (1+\xi)^2 \end{bmatrix} |J| d\xi = \frac{\rho AL}{8} \begin{bmatrix} (\xi + \frac{\xi^3}{3} - \xi^2) & (\xi - \frac{\xi^3}{3}) \\ (\xi - \frac{\xi^3}{3}) & (\xi + \frac{\xi^3}{3} + \xi^2) \end{bmatrix}_{-1}^{+1}$$

$$\Rightarrow m = \frac{\rho AL}{8} \begin{bmatrix} \frac{8}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{8}{3} \end{bmatrix} = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(4) Obtain also the mass matrix of a two-node, linear displacement element with a variable cross-sectional area that varies from A1 to A2.

$$A = N_1 A_1(\xi) + N_2 A_2(\xi)$$

$$m = \int N^T N \rho dV = \rho A \int N(\xi)^T N(\xi)|J|d\xi, where N = \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix}$$

$$= \frac{\rho AL}{16} \int_{-1}^{+1} \begin{bmatrix} (1-\xi)^2 & (1-\xi^2) \\ (1-\xi^2) & (1+\xi)^2 \end{bmatrix} ((1-\xi)A_1 + (1+\xi)A_2)d\xi$$

$$= \frac{\rho AL}{16} \int_{-1}^{+1} A_1 \begin{bmatrix} (1-\xi)^3 & (1-\xi^2)(1-\xi) \\ (1-\xi^2)(1-\xi) & (1+\xi)^2(1-\xi) \end{bmatrix} + A_2 \begin{bmatrix} (1-\xi)^2(1+\xi) & (1-\xi^2)(1+\xi) \\ (1-\xi^2)(1+\xi) & (1+\xi)^3 \end{bmatrix} d\xi$$

$$= \frac{\rho AL}{16} \int_{-1}^{+1} A_1 \begin{bmatrix} 1-3\xi+3\xi^2-\xi^3 & 1-\xi-\xi^2+\xi^3 \\ 1-\xi-\xi^2+\xi^3 & 1+\xi-\xi^2-\xi^3 \end{bmatrix} + A_2 \begin{bmatrix} 1-\xi-\xi^2+\xi^3 & 1+\xi-\xi^2-\xi^3 \\ 1+\xi-\xi^2-\xi^3 & 1+3\xi+3\xi^2+\xi^3 \end{bmatrix} d\xi$$

$$= \frac{\rho AL}{16} \left[A_1 \begin{bmatrix} \xi - \frac{3\xi^2}{2} + \xi^3 - \frac{\xi^4}{4} & \xi - \frac{\xi^2}{2} - \frac{\xi^3}{3} + \frac{\xi^4}{4} \\ \xi - \frac{\xi^2}{2} - \frac{\xi^3}{3} + \frac{\xi^4}{4} & \xi + \frac{\xi^2}{2} - \frac{\xi^3}{3} - \frac{\xi^4}{4} \end{bmatrix} + A_2 \begin{bmatrix} \xi - \frac{\xi^2}{2} - \frac{\xi^3}{3} + \frac{\xi^4}{4} & \xi + \frac{\xi^2}{2} - \frac{\xi^3}{3} - \frac{\xi^4}{4} \\ \xi + \frac{\xi^2}{2} - \frac{\xi^3}{3} - \frac{\xi^4}{4} & \xi + \frac{3\xi^2}{2} + \xi^3 + \frac{\xi^4}{4} \end{bmatrix} \right]_{-1}^{+1}$$

$$= A_1 \rho L \begin{bmatrix} 1/4 & 1/12 \\ 1/12 & 1/12 \end{bmatrix} + A_2 \rho L \begin{bmatrix} 1/12 & 1/12 \\ 1/12 & 1/4 \end{bmatrix} = \rho L \begin{bmatrix} \frac{3A_1+A_2}{12} & \frac{A_1+A_2}{12} \\ \frac{A_1+A_2}{12} & \frac{A_1+3A_2}{12} \end{bmatrix}$$

(5) A uniform two-node bar element is allowed to move in a 3D space. The nodes have only translational d.o.f. What is the diagonal mass matrix of the element?

Sol. 5 For 3D problem,

$$m = \frac{\rho AL}{2} I_6 = \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Since, the there is only translational d.o.f, so only diagnol element will be present while all only elements of the matrix will be o as there is no rotational d.o.f. Further, the matrix is a 6x6 matrix due to the fact there are 2 nodes with each node having 3 d.o.f(translational).