MASTER'S DEGREE NUMERICAL METHODS IN ENGINEERING

COMPUTATIONAL STRUCTURAL MECHANICS AND DYNAMICS

Homework 9: Axisymmetric shells

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Assignment 9.1

Describe in extension how can be applied a non-symmetric load to this formulation.

Assignment 9.2

Using thin beams formulation, describe the shape of the $B^{(e)}$ matrix and comment the integration rule.

1 Assignment 9.1

The application of non-symmetric loads into a formulation of axisymmetry shells require the utilization of *semi-analytical finite element processes*, as reported in section 9.5 Zienkiewicz [1].
The resolution of this problem is obtained by extending the plates solution, therefore firstly an introduction to the axisymmetry plates under non-symmetrical loads is presented. To this end Fourier expansions introduced and loads in the problem are divided into symmetric and non-symmetric parts. Also a three-dimensional scheme is considered for displacements given the loss of symmetry. Moreover, the loads are represented under schemes such as those presented in Figure 1.

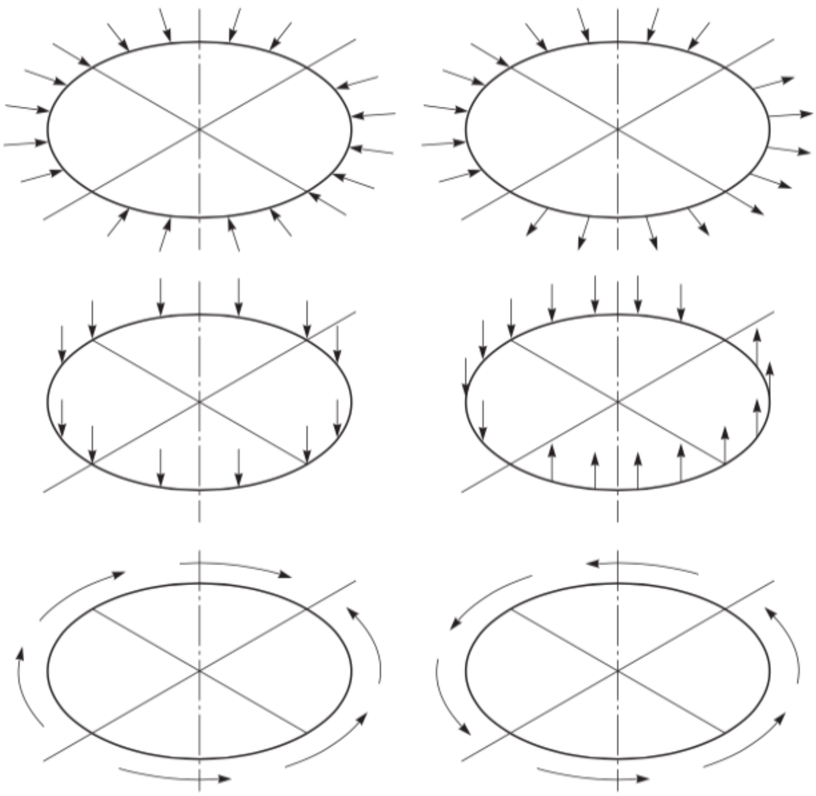

Figure 1: Considering non-symmetric loads in axisymmetric structures. Left: Non-symmetric. Right: Symmetric. [1]

Under this conditions the strains have to be treated as three-dimensional as:

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \\ \gamma_{z\theta} \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} u_r \\ w_z \\ [u + v_\theta]/r \\ u_z + w_r \\ u_z + w_\theta/r \\ u_\theta/r + v_r - v/r \end{bmatrix}$$

Using the above expressions, the map of displacements in three dimensions and describing them using Fourier series, an expression for $\underline{B_i^l}$ is written as:

$$\underline{B_i^l} = \begin{bmatrix} N_{i,r}cos(l \cdot \theta) & 0 & 0 \\ 0 & 0 & N_{i,z}cos(l \cdot \theta) \\ N_i/r \cdot cos(l \cdot \theta) & l \cdot N_i/r \cdot cos(l \cdot \theta) & 0 \\ N_{i,z}cos(l \cdot \theta) & 0 & N_{i,r}cos(l \cdot \theta) \\ 0 & N_{i,z}sin(l \cdot \theta) & -l \cdot N_i/r \cdot sin(l \cdot \theta) \\ -l \cdot N_i/r \cdot sin(l \cdot \theta) & (N_{i,r} - N_i/r)sin(l \cdot \theta) & 0 \end{bmatrix}$$

where the l represent the harmonic modes considered in the Fourier series.

As seen from the matrix, a purely axisymmetric behaviour can be recovered for $l = 0$. Then, to solve the problem the loads have to be described using Fourier series as well having the scheme showed in the equation below. Its worth mentioning that when $l = 0$ the middle term ($\overline{T}$) is zero, and the equation is constant in θ.

$$f_i^l = \int_0^{2\pi} \begin{bmatrix} \overline{R}^l \cdot cos^2(l \cdot \theta) \\ \overline{T}^l \cdot sin^2(l \cdot \theta) \\ \overline{Z}^l \cdot cos^2(l \cdot \theta) \end{bmatrix} d\theta$$

In this way, with every equation expressed in terms of the Fourier series, the *semi-analytical finite element processes* can be solved.

Axisymmetric shells under non-symmetrical loads: Considering the aforementioned integration over θ and using l order of Fourier series to interpolate the symmetric and anti-symmetric loads (using the idea in Figure 1), the equations below can be used to solve these axisymmetric shells under non-symmetrical lloads.

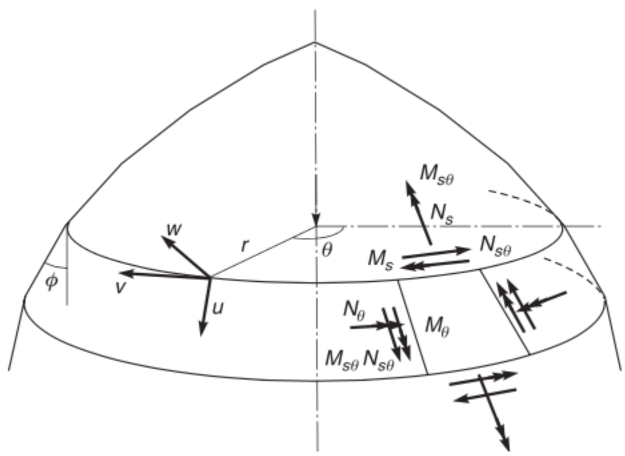

Figure 2: Considering non-symmetric loads in axisymmetric shells. [1]

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \gamma_{s\theta} \\ \chi_s \\ \chi_\theta \\ \chi_{s\theta} \end{bmatrix} = \begin{bmatrix} u_s \\ v,_\theta /r + (ucos(\phi) - wsin(\phi))/r \\ u,_\theta /r + v,_s - vcos(\phi)/r \\ -w,_{ss} \\ -w,_{\theta\theta} /r^2 - w,_s cos(\phi)/r + v,_\theta sin(\phi)/r \\ 2 \cdot (-w,_{s\theta} /r - w,_\theta cos(\phi)/r^2 + v,_s sin(\phi)/r - v\ sin(\phi)cos(\phi)/r^2) \end{bmatrix}$$

Then the *stresses* will be:

$$\underline{\sigma} = \begin{bmatrix} N_s \\ N_\theta \\ N_{s\theta} \\ M_s \\ M_\theta \\ M_{s\theta} \end{bmatrix}$$

2 Assignment 9.2

Using the Reissner-Mindlin thin beam formulation shear locking effects arise as also did for beam elements. The simpler procedure to avoid shear locking is the reduced/selective quadrature. The reduced integration of the membrane stiffness terms improves the in-plane behaviour and it also eliminates membrane locking that may appear in same special cases. The simplest case for solving this problem is to use a 2-noded strip with a single point quadrature. In Figure 3 shows the generalized strain matrix computed at the element mid-point [2].

$$[\mathbf{K}'^{ll}_{ij}]^{(e)} = \frac{ba^{(e)}}{2}[\bar{\mathbf{B}}^l_i]^T \hat{\bar{\mathbf{D}}}' \bar{\mathbf{B}}^l_j$$

$\bar{(\cdot)} \equiv$ values at the strip mid-point.

$$\bar{\mathbf{B}}^l_i = \left\{ \begin{array}{c} \bar{\mathbf{B}}^l_{m_i} \\ ---- \\ \bar{\mathbf{B}}^l_{b_i} \\ ---- \\ \bar{\mathbf{B}}^l_{s_i} \end{array} \right\} = \begin{bmatrix} A_iC & 0 & A_iS & 0 & 0 & 0 \\ 0 & -\frac{\gamma}{2} & 0 & 0 & 0 & 0 \\ \frac{\gamma}{2}C & A_i & \frac{\gamma}{2}S & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & -A_i & 0 \\ 0 & 0 & 0 & -\frac{\gamma}{2}C & 0 & -\frac{\gamma}{2}S \\ 0 & 0 & 0 & A_iC & -\frac{\gamma}{2} & -A_iS \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_iS & 0 & A_iC & 0 & \frac{1}{2} & 0 \\ -\frac{\gamma}{2}S & 0 & -\frac{\gamma}{2}C & -\frac{1}{2}C & 0 & -\frac{1}{2}S \end{bmatrix}$$

$$A_i = \frac{(-1)^i}{a^{(e)}} \;;\; S = \sin\phi^{(e)} \;;\; C = \cos\phi^{(e)} \;;\; \gamma = \frac{l\pi}{b}$$

Figure 3: Two-noded flat shell strip element. Stiffness matrix computed using one single point reduced integration [2].

References

[1] O. C. Zienkiewicz and R. Taylor, *The Finite Element Method. Volume Two: Solid Mechanics.* Butterworth-Heinemann, 2000.

[2] E. Oñate, *Structural analysis with the finite element method. Linear statics. Volume Two: Beams, Plates and Shells.* Springer, 2013.