Assignment 9

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May 14, 2020

(1) Rewriting the formula from slide 7:

$$m\ddot{u} + ku = F \tag{1}$$

The displacement u is divided into a homogenous and particular solutions where:

$$u = u_h + u_p \tag{2}$$

u_h is solved for $F = 0$, and u_p is a constant. Each are replaced in equation (1) and solved, and then replaced in equation (2). The solution for each equation are the following:

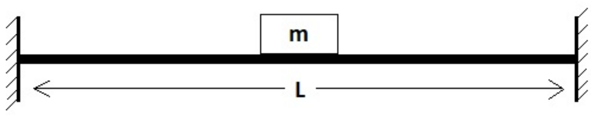
$$\begin{aligned} u_h &= C_1 sin(\omega t) + C_2 cos(\omega t) \\ u_p &= \frac{F}{k} \\ \rightarrow u &= C_1 sin(\omega t) + C_2 cos(\omega t) + \frac{F}{k} \end{aligned}$$

taking initial conditions as $u(0) = u_0 \quad \& \quad \dot{u}(0) = v_0$, we get $C_1 = \frac{v_0}{\omega} \quad \& \quad C_2 = u_0 - \frac{F}{k}$. Therefore the solution is:

$$u = \frac{v_0}{\omega} sin(\omega t) + \left(u_0 - \frac{F}{k}\right) cos(\omega t) + \frac{F}{k}$$

The natural frequency remains unchanged ($\omega = \sqrt{k/m}$), and only the amplitude of the oscillations is changed, because F can be considered a constant displacement of the system.

(2) The following is the representation of the given in problem (2):



The effective stiffness of a clamped beam with a force at the center is:

$$k = \frac{192EI}{L^3}$$

Assuming it is a square beam, $I = A^2/12$ therefore the stiffness is:

$$k = \frac{16EA^2}{L^3}$$

Therefore the natural frequency is simply:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{16EA^2}{mL^3}}$$

(3) The equation from slide 20 is the following:

$$m = \int \mathbf{N}^T \mathbf{N} \rho dV = \rho A \int \mathbf{N}^T \mathbf{N} dx$$

Where $\mathbf{N}$ is a vector of linear shape functions:

$$\mathbf{N} = \begin{bmatrix} \frac{L-x}{L} & \frac{x}{L} \end{bmatrix}$$

The multiplication of the vectors is the following:

$$\mathbf{N}^T \mathbf{N} = \begin{bmatrix} \frac{L-x}{L} \\ \frac{x}{L} \end{bmatrix} \begin{bmatrix} \frac{L-x}{L} & \frac{x}{L} \end{bmatrix} = \begin{bmatrix} \frac{L^2-2Lx+x^2}{L^2} & \frac{Lx-x^2}{L^2} \\ \frac{Lx-x^2}{L^2} & \frac{x^2}{L^2} \end{bmatrix}$$

Therefore, for each element and x between 0 and L, the integration of each term multiplied by ρA:

$$m = \frac{\rho A}{L^2} \begin{bmatrix} L^3 - L^3 + L^3/3 & L^3/2 - L^3/3 \\ L^3/2 - L^3/3 & L^3/3 \end{bmatrix} = \begin{bmatrix} \frac{\rho AL}{3} & \frac{\rho AL}{6} \\ \frac{\rho AL}{6} & \frac{\rho AL}{3} \end{bmatrix}$$

(4) The same formulation will be followed but only changing the area for the integration while the area has the following formula:

$$A(x) = A_1 + \frac{x}{L}(A_2 - A_1)$$

The integration for the first term (m_1 with A_1 only)is still the same, but the integration for the second term will change, and will be calculated as followed:

$$m = m_1 + m_2$$

$$m_2 = \rho \frac{A_2 - A_1}{L} \int \mathbf{N}^T \mathbf{N} x dx$$

$$\mathbf{N}^T \mathbf{N} x = x \begin{bmatrix} \frac{L-x}{L} \\ \frac{x}{L} \end{bmatrix} \begin{bmatrix} \frac{L-x}{L} & \frac{x}{L} \end{bmatrix} = \begin{bmatrix} \frac{xL^2-2Lx^2+x^3}{L^2} & \frac{Lx^2-x^3}{L^2} \\ \frac{Lx^2-x^3}{L^2} & \frac{x^3}{L^2} \end{bmatrix}$$

$$m_1 = m(part(3)) \quad (for \quad A = A_1)$$

$$m_2 = \frac{\rho(A_2 - A_1)}{L^3} \begin{bmatrix} L^4/2 - 2L^4/3 + L^4/4 & L^4/3 - L^4/4 \\ L^4/3 - L^4/4 & L^4/4 \end{bmatrix} = \rho L(A_2 - A_1) \begin{bmatrix} 1/12 & 1/12 \\ 1/12 & 1/4 \end{bmatrix}$$

$$m = \frac{\rho L}{12} \begin{bmatrix} 3A_1 + A_2 & 2A_1 + A_2 \\ 2A_1 + A_2 & A_1 + A_2 \end{bmatrix}$$

(5) Slide 22 will be followed as an example. Each node has three d.o.f therefore the diagonal mass matrix of the element is a 6×6 matrix for a two noded element. the total mass of the element is ρAL while ρ is the density, A is the cross-sectional area and L the length of the element, and divided to each element will half of its value, therefore the diagonal mass matrix is:

$$m = \frac{\rho AL}{2}\mathbf{I}_6$$