Computational Solid Mechanics

Assignment 9: Revolution Shells

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SUMMARY

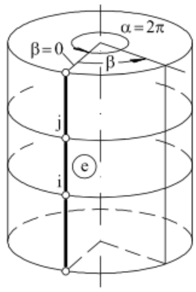
Assignment 9.1

Describe in extension how can be applied a non-symmetric load on this formulation

The axisymmetric shell formulation can be extended for an analysis under arbitrary loading. The main idea is to apply the finite strip method.

This method consists on representing the loads and the displacements by means of Fourier series in the direction of circumference. Therefore, the Fourier expansions are written in term of the angle β. In this way, it is possible to assemble a stiffness matrix for every harmonic term and solve the displacements.

The following figure shows an axisymmetric shell discretized in circular strips.



b)

The displacements are expanded in Fourier series along the circumferential direction splitting the displacement field in symmetric and anti-symmetric components.

$$\mathbf{u}' = \sum_{l=0}^{m} \sum_{i=1}^{n} \mathbf{N}_i (\bar{\mathbf{S}}^l \bar{\mathbf{a}}_i'^l + \bar{\bar{\mathbf{S}}}^l \bar{\bar{\mathbf{a}}}_i'^l)$$

Being $\boldsymbol{u}'$

$$\mathbf{u}'(s,\beta) = [u_0', v_0', w_0', \theta_s, \theta_t]^T$$

And S^l are the trigonometric functions of the *l-th* harmonic term, and a_i^l is the modal displacement amplitude vector.

$$\mathbf{S}^l(y) = \begin{bmatrix} S^l & & & & \\ & C^l & & & \mathbf{0} \\ & & S^l & & \\ & & & S^l & \\ & \mathbf{0} & & & C^l \end{bmatrix}$$

The previous equation shows the expression for the antisymmetric trigonometric function. The symmetric one is obtained simply swapping S^l and C^l.

The loads are expanded in Fourier series analogously.

$$\mathbf{t} = \sum_{l=0}^{m} (\bar{\mathbf{S}}^l \bar{\mathbf{t}}^l + \bar{\bar{\mathbf{S}}}^l \bar{\bar{\mathbf{t}}}^l)$$

The computation of the symmetric and anti-symmetric solutions is carried out separately to simplify the analysis.

The local stiffness matrix for an axisymmetric strip element is

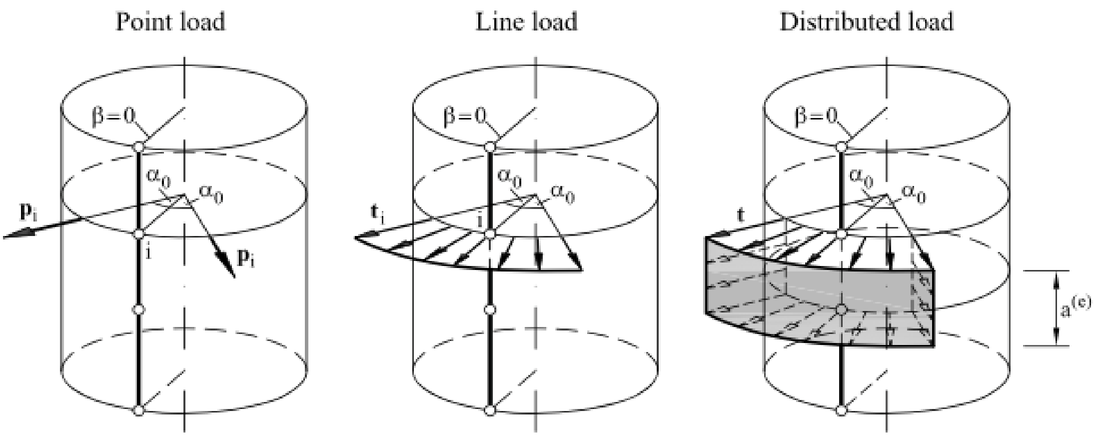
$$[\mathbf{K}_{ij}^{\prime ll}]^{(e)} = C \int_{a^{(e)}} [\mathbf{B}_i^{\prime l}]^T \hat{\mathbf{D}}' \mathbf{B}_j^{\prime l} r ds$$

Where

$$C = \begin{cases} 2\pi & \text{for } \; l = 0 \\ \pi & \text{for } \; l \neq 0 \end{cases}$$

The computation of the local strain matrix is done and its transformation to the global axes is carried out.

The expression for the equivalent nodal force vector for the 2-noded strip for different types of loads is presented below



Point load	Line load	Distributed load
$[\mathbf{f}_i^0]^{(e)} = 2\mathbf{p}_i$	$2 r_i \alpha_0 \mathbf{t}_i$	$\alpha_0 a^{(e)} \bar{r}^{(e)} \mathbf{t}$
$[\mathbf{f}_i^l]^{(e)} = 2 \cos l\pi \cos l\alpha_0 \mathbf{p}_i$	$\frac{2 r_i}{l} \sin l\alpha_0 \cos l\pi \mathbf{t}_i$	$\frac{a^{(e)} \bar{r}^{(e)}}{l} \sin l\alpha_0 \cos l\pi \mathbf{t}$

Self weight $[\mathbf{f}_i^l]^{(e)} = 0$; $[\mathbf{f}_i^0]^{(e)} = \pi a^{(e)} \bar{r}^{(e)} \rho g [0,0,-1,0,0,0]^T$

$\bar{r}^{(e)}$ = radius of the strip mid-point

NOTE: All the procedure to implement arbitrary loads in the axisymmetric shell formulation with 2-noded strip has been extracted from [1].

Assignment 9.2

Using thin beams formulation, describe the shape of $B^{(e)}$ matrix and comment the integration rule

Assuming the Kirchhoff theory, the normal to the generatrix remains straight and orthogonal after deformation. Therefore, the normal rotation coincides with the slope of the generatrix at each point.

With this assumption the effect of the traverse shear deformation is neglected. For that reason, only membrane and bending strains are considered.

$$\hat{\boldsymbol{\varepsilon}}'_m = \begin{Bmatrix} \dfrac{\partial u'_0}{\partial s} - \dfrac{w'_0}{R_s} \\ \dfrac{u'_0 \cos\phi - w'_0 \sin\phi}{r} \end{Bmatrix}; \quad \hat{\boldsymbol{\varepsilon}}'_b = \begin{Bmatrix} \dfrac{\partial^2 w'_0}{\partial s^2} + \dfrac{\partial}{\partial s}(\dfrac{u'_0}{R_s}) \\ \dfrac{\cos\phi}{r}\left(\dfrac{\partial w'_0}{\partial s} + \dfrac{u'_0}{R_s}\right) \end{Bmatrix}$$

Since the second derivative of normal displacement appears, C^1 continuity is needed for the approximation of w_0. However, a C_0 Lagrange approximation can be employed for the tangential displacement.

The tangential displacement is interpolated as

$$u'_0 = \sum_{i=1}^{2} N_i^u \, u'_{o_i} \qquad \text{with} \qquad N_i^u = \frac{1+\xi\xi_i}{2}$$

And the normal one with a C^1 continuous approximation

$$w'_0 = \sum_{i=1}^{2} \left[N_i^w \, w'_{o_i} + \bar{N}_i^w \left(\frac{\partial w'_0}{\partial s}\right)_i \right]$$

Being N_i^w the cubic 1D Hermite shape functions.

The local generalized strain matrix is expressed as

$$\mathbf{B}'_i = \begin{Bmatrix} \mathbf{B}'_{m_i} \\ --- \\ \mathbf{B}'_{b_i} \end{Bmatrix} = \left[\begin{array}{ccc} \dfrac{\partial N_i^u}{\partial s} & 0 & 0 \\ \dfrac{N_i^u \cos\phi}{r} & \dfrac{-N_i^w \sin\phi}{r} & \dfrac{-\bar{N}_i^w \sin\phi}{r} \\ \hline 0 & \dfrac{\partial^2 N_i^w}{\partial s^2} & \dfrac{\partial^2 \bar{N}_i^w}{\partial s^2} \\ 0 & \dfrac{\cos\phi}{r}\dfrac{\partial N_i^w}{\partial s} & \dfrac{\cos\phi}{r}\dfrac{\partial \bar{N}_i^w}{\partial s} \end{array}\right]$$

NOTE: Formulation extracted from [1]

Comments on integration rule

The Gaussian quadrature is the best choice to integrate numerically the stiffness matrix and is generally recommended the use of two gauss points to obtain good results [2]. However, Grafton and Strone [3] suggested on their work an explicit formula for the stiffness matrix based on a single average value of the integrand (one-point Gaussian quadrature). This reduced integration is good enough for the regions with low stress gradients, but near the boundaries, the high gradients inside the boundary layers need local mesh refinements to achieve good results. The results presented by the authors cited above get good accuracy by means of this local refinement.

References

[1] E. Oñate, "Structural Analysis with the Finite Element Method Linear Statics Volume 2. Beams, Plates and Shells," in *Springer*, 2013, pp. 564-568 , 676–705.

[2] O. C. Zienkiewicz and R. L. Taylor, *The Finite Element Method Volume 2: Solid Mechanics*. 1981.

[3] P. E. Grafton and D. R. Strome, "Analysis of axisymmetrical shells by the direct stiffness method," *AIAA J.*, vol. 1, no. 10, pp. 2342–2347, 1963.