ASSIGNMENT 7

Computational Structural Mechanics and Dynamics

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Background:

In continuum mechanics, plate theories are mathematical descriptions of the mechanics of flat plates that draws on the theory of beams. Plates are defined as plane structural elements with a small thickness compared to the planar dimensions. Kirchhoff plate elements have obviously limitations to reproduce the behaviour of thick plates. Reissner-Mindlin plate elements are superior as they are applicable to thick and thin situation. Reissner-Mindlin plate theory is very adequate for studying composite laminate plates for which shear deformation effects are important. Reissner-Mindlin plate theory is the basis for systematically deriving C_0 continuous plate elements which include transverse shear strain effects. Practically, the only drawback of Reissner-Mindlin plate elements is the appearance of shear locking for thin plate situations. The reduced integration of the transverse shear stiffness terms is a simple and efficient procedure for eliminating shear locking, although it can introduce spurious mechanisms which can pollute the solution in some cases. The assumed transverse shear strain technique is a more consistent approach for designing robust locking-free plate elements.

The normal orthogonality condition only holds for thin plates (thickness/ average side ratio:$t/L \leq 0.05$). For moderately thick ($0.05 \leq t/L \leq 0.1$) and very thick ($t/L \geq 0.10$) plates, the distortion of the normal during deformation increases. Reissner-Mindlin theory represents a better approximation to the actual deformation of the plate in these cases.

(A) For the given problem only a quarter of the plate is discretized due to symmetry.

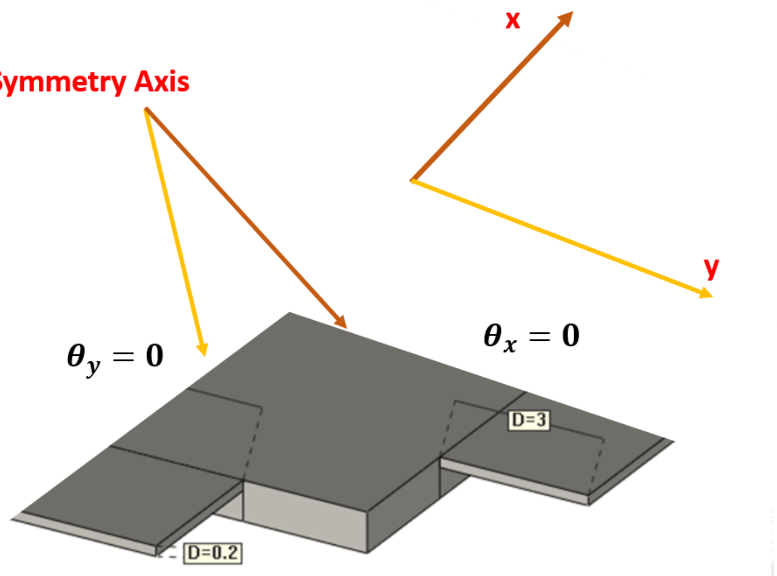


Figure: 1 Quarter of the plate

Boundary conditions along the Outer edges:

- If the outer edges are clamped $w = \theta_x = \theta_y = 0$ is employed
- Else, If the outer edges are simply supported:
    1. Hard support ($w = \theta_s = 0$)
    2. Soft support ($w = 0$)

Along the line of symmetry parallel to the X axis, $\theta_y = 0$ and along the line of symmetry parallel to the Y axis, $\theta_x = 0$.

As the normal orthogonality condition only holds for thin plates (thickness/ average side ratio:$t/L \leq 0.05$). Here the t/L can be found to be 0.05 and 0.08. In this case, the distortion of the normal during deformation increases. Here, shear effects are very important. Reissner-Mindlin plate elements are superior as they are applicable to thick and thin situation. Reissner-Mindlin plate theory is very adequate for studying composite laminate plates for which shear deformation effects are important.

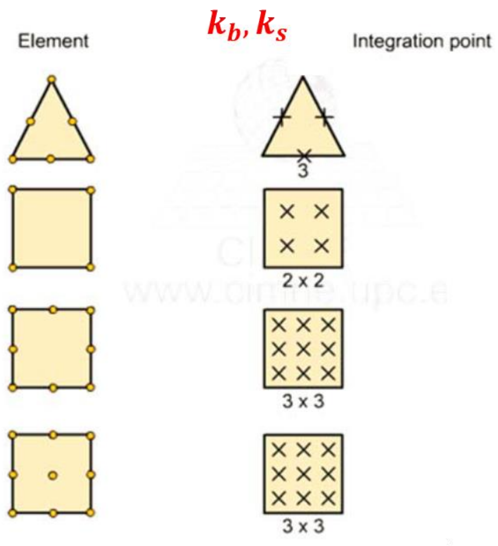
Theory:-

Reissner-Mindlin plate bending theory (Full Integration Rule)

- In the points belonging to the middle plane (z = 0), u = v = 0. In other words, the points on the middle plane only move vertically.
- The points along a normal to the middle plane have the same vertical Displacement (i.e. the thickness does not change during deformation).
- The normal stress $\sigma_z$is negligible (plane stress assumption).
- A straight line normal to the un-deformed middle plane remains straight but not necessarily orthogonal to the middle plane after deformation.

Element	**Quadrature**	Quadrature for Integration of	
		${k_b}^e$	${k_s}^e$
Q4	FULL	2*2	2*2
QS8,QL9,QH9,QHG,QHET	FULL	3*3	3*3
QS12,QL16	FULL	4*4	4*4

The following elements can be used to obtain reasonably good results.



(B) For the given problem only a quarter of the plate is discretized due to symmetry.

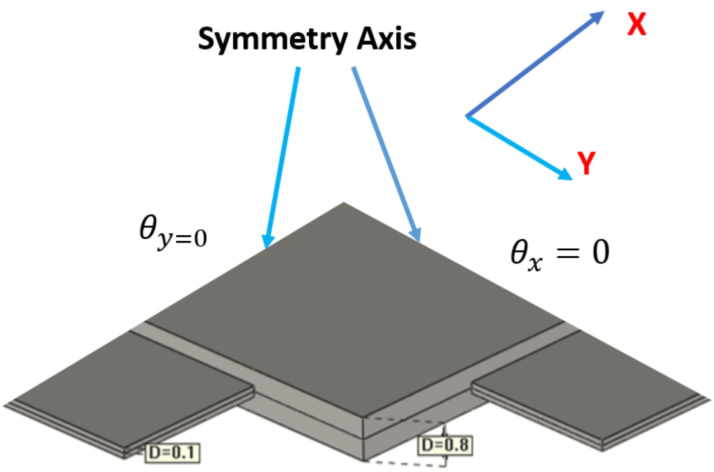


Figure 2: Quarter of the plate (Analysis domain)

Boundary conditions along the Outer edges:

- If the outer edges are clamped $w = \theta_x = \theta_y = 0$ is employed
- Else, If the outer edges are simply supported:
  3. Hard support ($w = \theta_s = 0$)
  4. Soft support ($w = 0$)

Along the line of symmetry parallel to the X axis, $\theta_y = 0$ and along the line of symmetry parallel to the Y axis, $\theta_x = 0$.

As the normal orthogonality condition only holds for thin plates (thickness/ average side ratio:$t/L \leq 0.05$). Here the t/L can be found to be 0.025 and 0.08. Here, both the shear and bending effects are very important. So, Reissner-Mindlin plate elements with the reduced integration scheme are used.

Theory:-

Reissner-Mindlin plate bending theory with the **reduced integration** of the transverse shear stiffness terms is used for this second case.

Assumptions:-

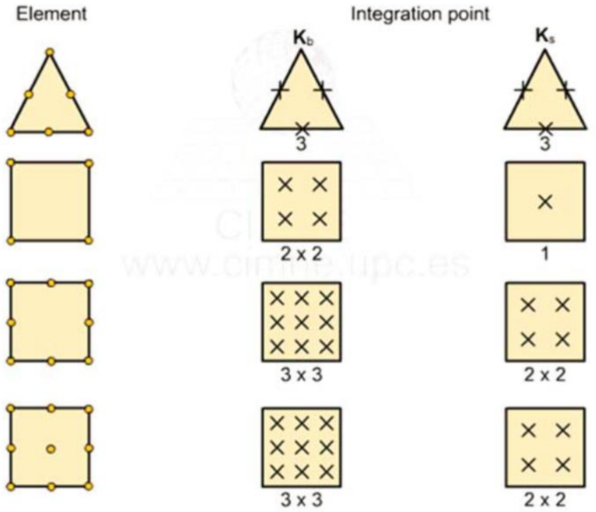
- In the points belonging to the middle plane (z = 0), u = v = 0, In other words, the points on the middle plane only move vertically.
- The points along a normal to the middle plane have the same vertical Displacement (i.e. the thickness does not change during deformation).
- The normal stress $\sigma_z$is negligible (plane stress assumption).
- A straight line normal to the un-deformed middle plane remains straight but not necessarily orthogonal to the middle plane after deformation.

Elements and integration rules:

Element	**Quadrature**	Quadrature for Integration of	
		${k_b}^e$	${k_s}^e$
Q4	Selective (S)	2*2	1*1
QS8,QL9,QH9,QHG,QHET	S	3*3	2*2
QS12,QL16	S	4*4	3*3

Table 1: Selective (S) quadrature's for various Reissner-Mindlin plate quadrilaterals

The elements showing reasonably better results have been presented in the table below.



Reduced integration can induce additional zero eigenvalues in the element stiffness matrix and hence originate new mechanisms in addition to the rigid body motions. These new mechanisms can or cannot propagate themselves within a mesh. This depends on their compatibility with adjacent elements and with the boundary conditions.

Patch test:

The patch test in the finite element method is a simple indicator of the quality of a finite element. The patch test uses a partial differential equation on a domain consisting from several elements set up so that the exact solution is known and can be reproduced, in principle, with zero error.

A broader definition of patch test (applicable to any numerical method, including and beyond finite elements) is any test problem having an exact solution that can, in principle, be exactly reproduced by the numerical approximation.

MZC element plate domains satisfies the patch test for rectangular plate elements. This ensures the convergence as the mesh is refined. Unfortunately the patch test is not satisfied for arbitrary quadrilateral shapes as the constant curvature criterion is violated in those situations. This limits the application of the MZC element plate domains that can be discretized into rectangular plate elements. However, in these cases it is an accurate element. We see that the MZC thin plate element, although it is incompatible, converges in its rectangular form.

The test:

In the investigated patch test, the displacements of the nodes at the boundary of the patch are prescribed. Then the values of the displacement at the internal nodes are computed and compared to the exact ones. A simple patch test of this type can be applied in thin plate elements in order to verify the good representation of rigid body displacements and the absence of spurious modes.

Case 1: The following displacement field is imposed at the boundary nodes:

$$w = \frac{20-3x-4y}{100} \qquad (1)$$

After solving the system of equations the internal DOFs must comply with Eq. (1) and the curvatures must be zero at each point in the patch.

Case 2: A similar type of patch test was devised for verifying the capability of the element for reproducing a constant curvature field. The test is based on imposing to the patch boundary nodes the quadratic displacement field:

$$w = \frac{3x^2+4y^2+2xy}{100} \qquad (2)$$

The numerical solution for the deflection at internal nodes must be in accordance with Eq. 2.Also, the curvature field must be constant at each point within the element.

Case 1 (On imposing to the patch boundary nodes the linear displacement field)

A. Rectangular MZC plate elements:

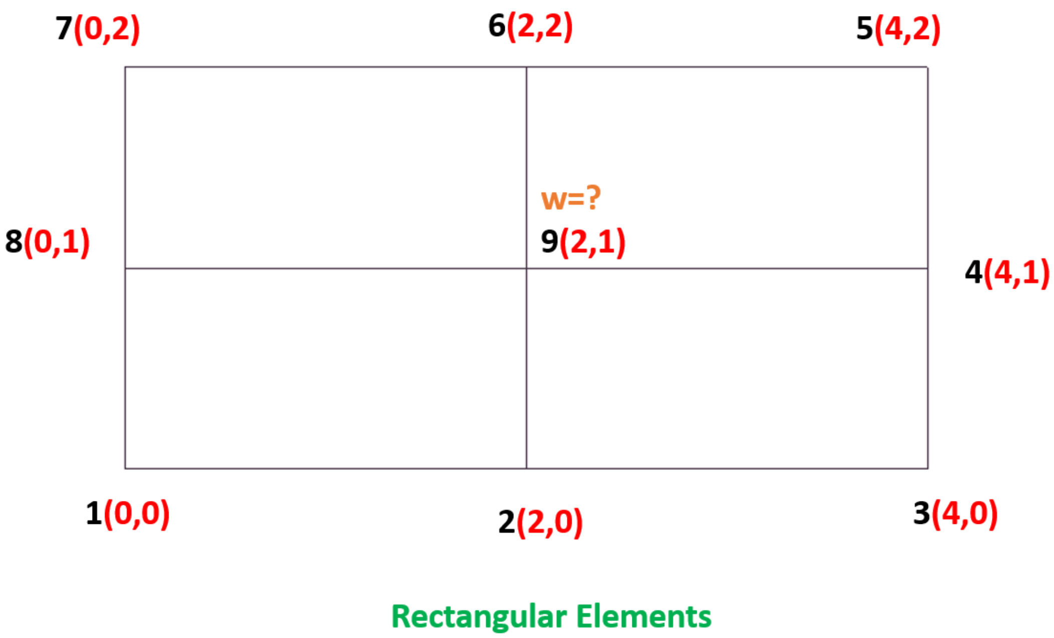


Figure 1: Considered rectangular elements for the Patch test

Node number (On the boundary)	Imposed Linear Displacement	Material Properties for MZC plate elements
1	0.2	E=2.1e11Pa $\gamma = 0.2$ Thickness=0.01m
2	0.14	
3	0.08	
4	0.04	
5	0	
6	0.06	
7	0.12	
8	0.16	

$\theta_x = -0.03;\ \theta_y = -0.04$

The exact solution sought at the node 9**(2, 1)** is 0.1m

The evaluated solution using the MZC plate elements for the patch test = 0.1m

So, the rectangular MZC plate elements for the on imposing to the patch boundary nodes the linear displacement field successfully passes the patch test.

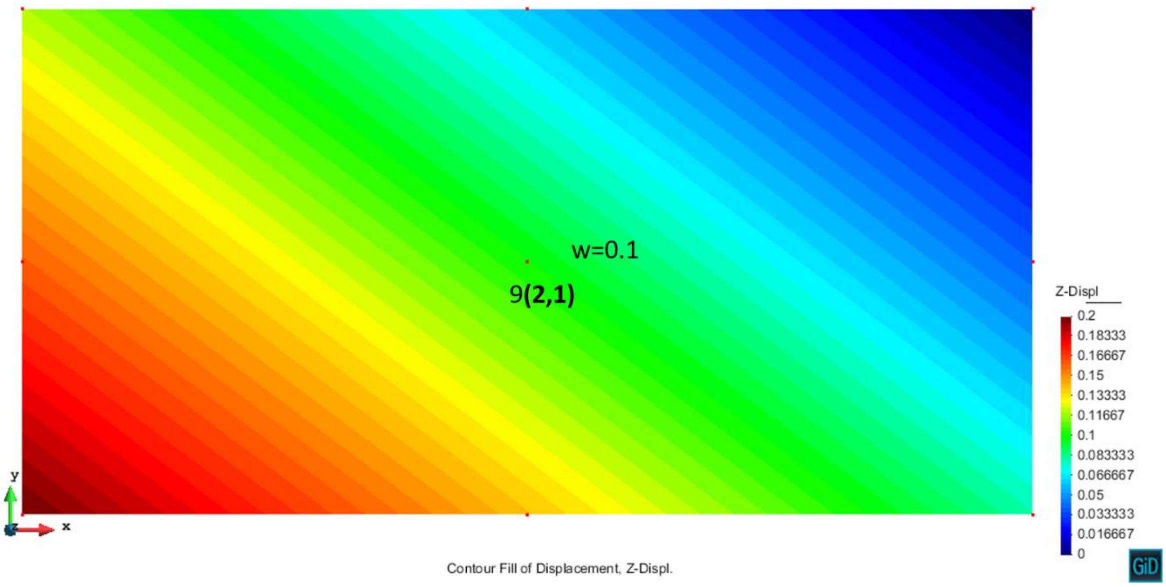


Figure 2: Solution of the displacement field at the internal node using rectangular MZC plate elements upon imposing to the patch boundary nodes the linear displacement field

B. Any other Quadrilateral MZC plate Elements:

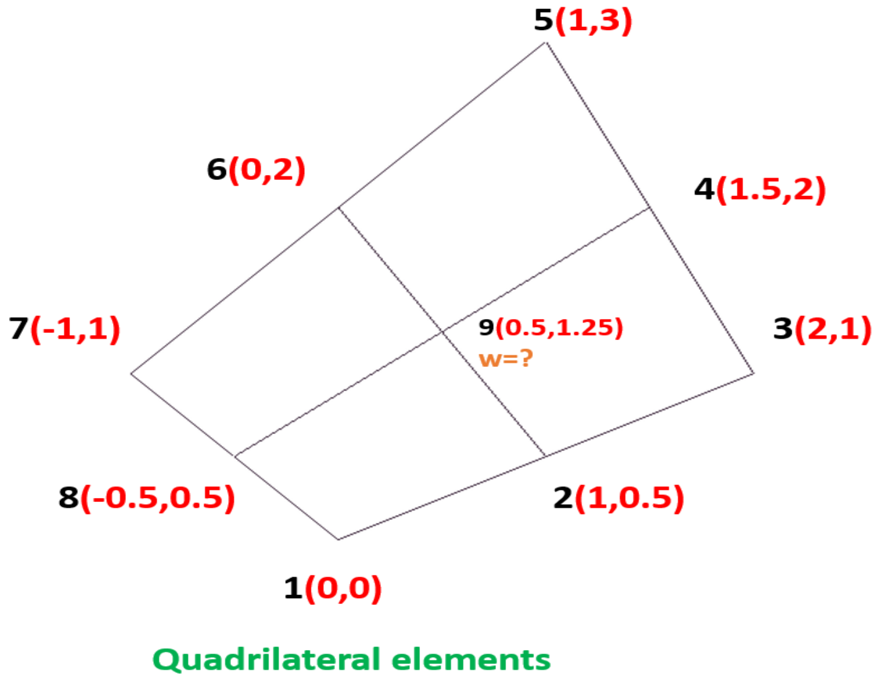


Figure 3: Quadrilateral Elements

Node number (On the boundary)	Imposed Linear Displacement	Material Properties for MZC plate elements
1	0.2	E=2.1e11Pa $\gamma = 0.2$ Thickness=0.01m
2	0.15	
3	0.1	
4	0.105	
5	0.05	
6	0.12	
7	0.19	
8	0.195	

$\theta_x = -0.03;\ \theta_y = -0.04$

The exact solution sought at the node 9**(0.5, 1.25)** is 0.135m

The evaluated solution using the MZC plate elements for the patch test = 0.13786m

So, all other quadrilateral MZC plate elements for the on imposing to the patch boundary nodes the linear displacement field does not successfully pass the patch test always. So, it is concluded that the non-conforming MZC element converges for the rectangular elements.

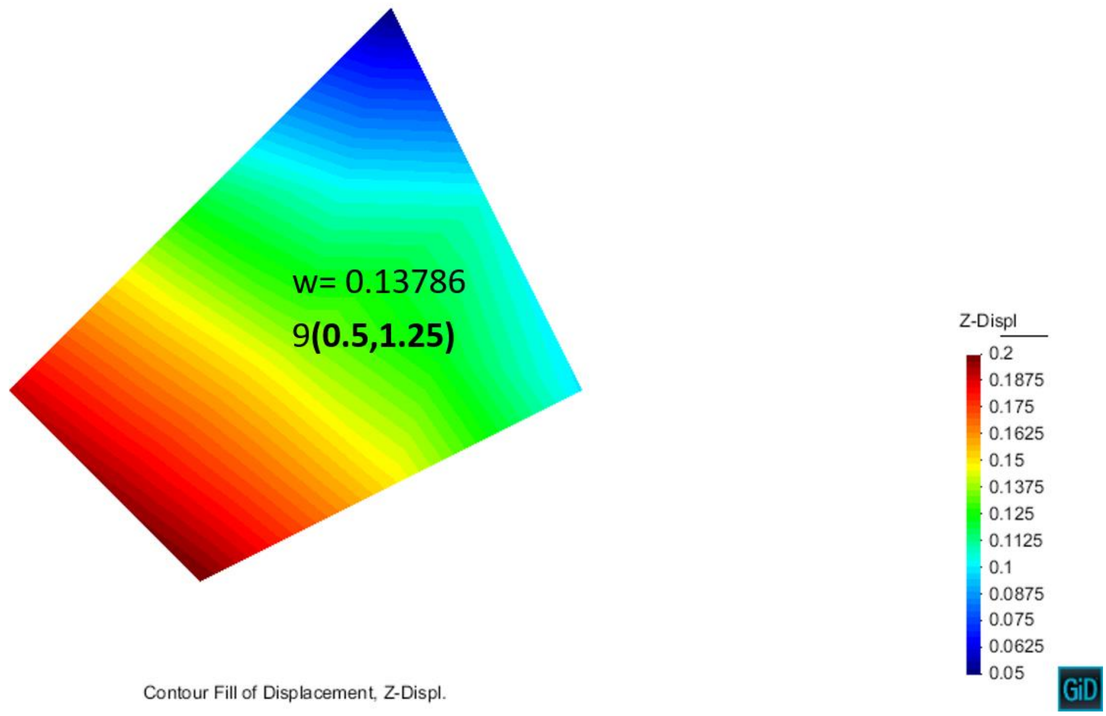


Figure 4: Solution of the displacement field at the internal node using a quadrilateral MZC plate elements upon imposing to the patch boundary nodes the linear displacement field

CASE 2: (On imposing to the patch boundary nodes the quadratic displacement field)

A. Rectangular MZC plate elements:

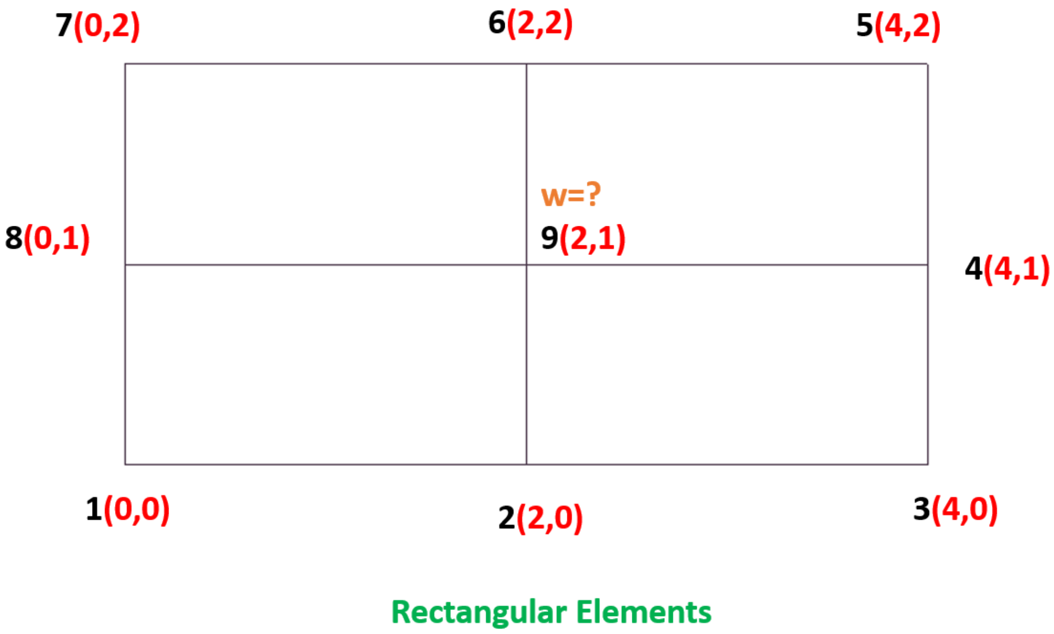


Figure 5: Considered rectangular elements for the Patch test

Node number (On the boundary)	Imposed quadratic Displacement	$\theta_x = (6x + 2y)/100$	$\theta_y = (2x + 8y)/100$	Material Properties for MZC plate elements
1	0	0	0	E=2.1e11Pa $\gamma = 0.2$ Thickness=0.01m
2	0.12	0.12	0.04	
3	0.48	0.24	0.08	
4	0.6	0.26	0.12	
5	0.8	0.28	0.24	
6	0.36	0.16	0.2	
7	0.16	0.04	0.16	
8	0.04	0.02	0.08	

The exact solution sought at the node 9**(2, 1)** is 0.2m

The evaluated solution using the MZC plate elements for the patch test = 0.2m

So, the rectangular MZC plate elements for the on imposing to the patch boundary nodes the quadratic displacement field passes the patch test.

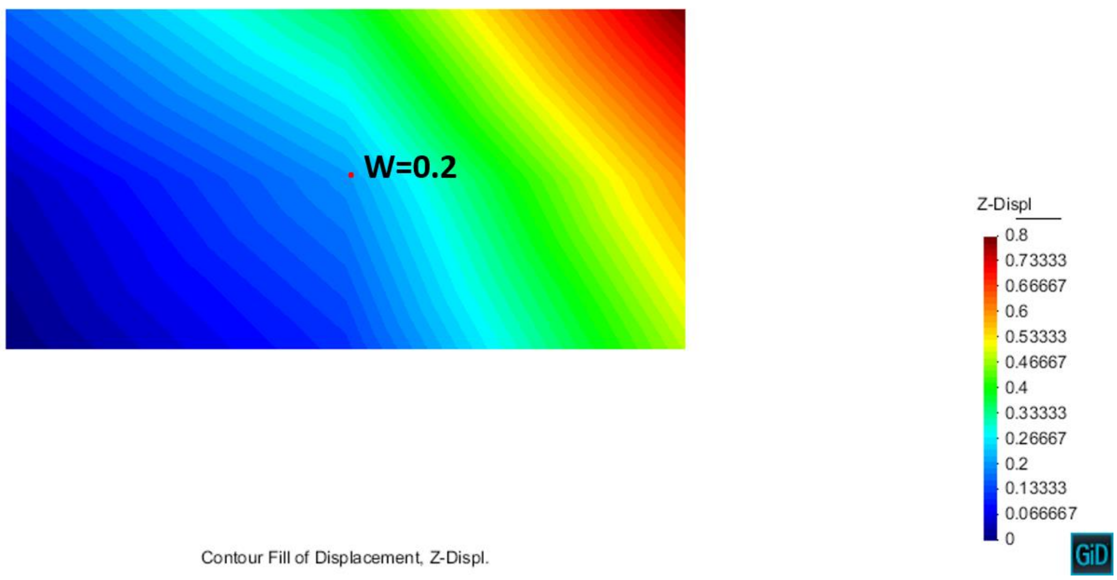
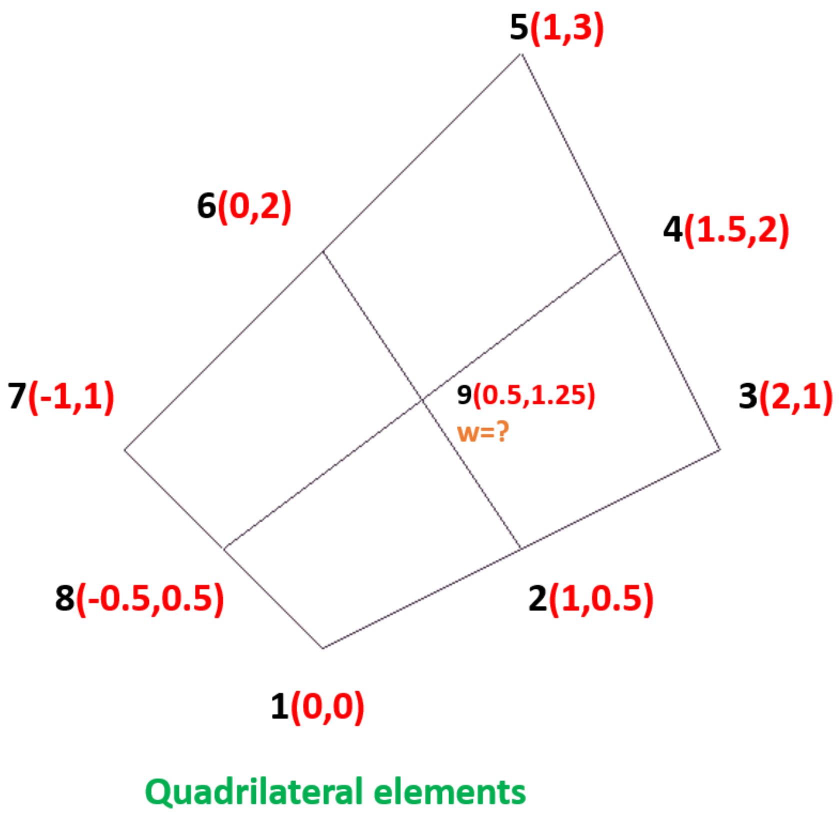


Figure 7: Solution of the displacement field at the internal node using rectangular MZC plate elements upon imposing to the patch boundary nodes the quadratic displacement field

B. Quadrilateral MZC plate elements:



Node number (On the boundary)	Imposed Quadratic Displacement	$\theta_x = (6x + 2y)/100$	$\theta_y = (2x + 8y)/100$	Material Properties for MZC plate elements
1	0	0	0	E=2.1e11Pa $\gamma = 0.2$ Thickness=0.01m
2	0.05	0.07	0.06	
3	0.2	0.1	0.12	
4	0.2875	0.13	0.19	
5	0.45	0.12	0.26	
6	0.16	0.04	0.16	
7	0.09	-0.04	0.06	
8	0.0225	-0.02	0.03	

The exact solution sought at the node 9**(2, 1)** is 0.0825m

The evaluated solution using the MZC plate elements for the patch test = 0.13513m

So, the quadrilateral MZC plate elements on imposing to the patch boundary nodes, the quadratic displacement field does not pass the patch test.

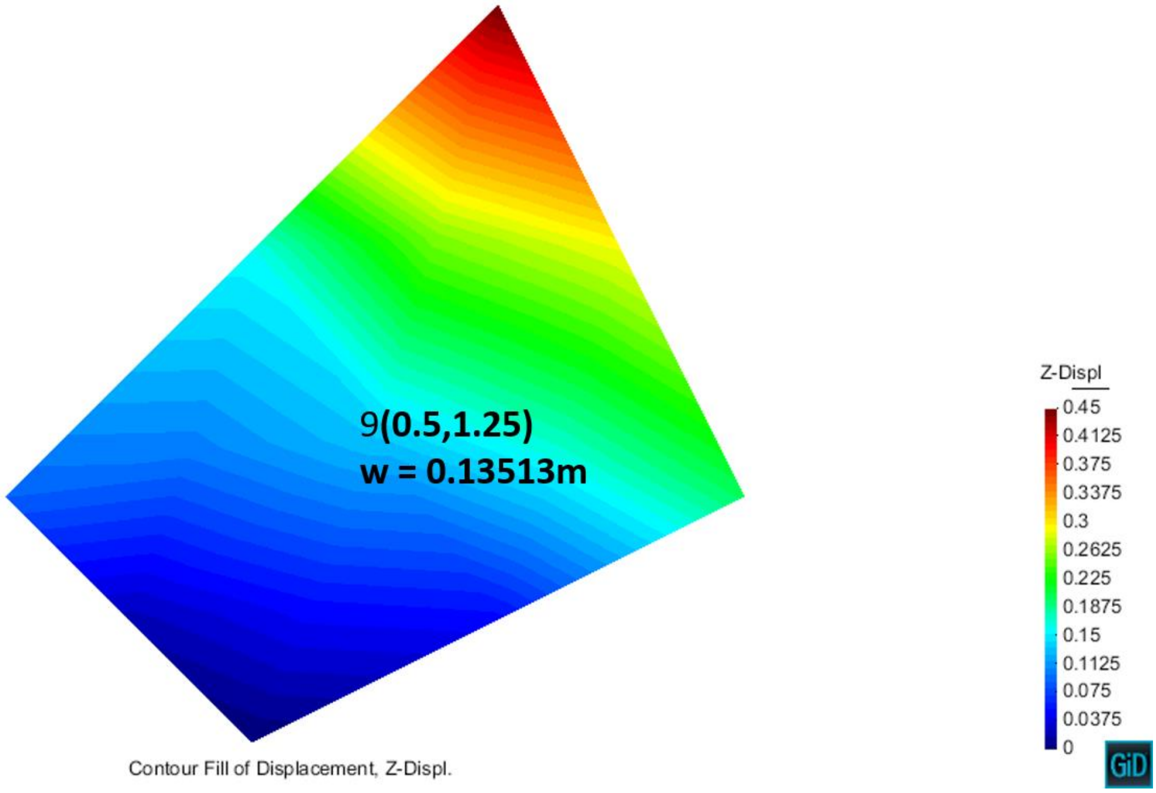


Figure 8: Solution of the displacement field at the internal node using quadrilateral MZC plate elements upon imposing to the patch boundary nodes the quadratic displacement field

It is found that on imposing to the patch boundary nodes the linear and quadratic displacement fields, only the rectangular MZC elements pass the patch test. So, the non-conforming MZC element converges for the rectangular elements. On the other hand, it does not converge for any other quadrilateral of random shape. In addition to these, it is found that on imposing to the patch boundary nodes the quadratic displacement field, the rectangular MZC plate elements does pass the patch test. So, in this case, the accuracy will be better and we do not need to use small elements.

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