Master on Numerical Methods in Engineering

Computational Structural Mechanics and Dynamics

Assignment 6

Bending of beams 1D

- Euler Bernoulli Beams Theory
- Timoshenko Beams Theory (Full and reduced integration)

Use of Matlab and GiD software

Mónica Ortega Castro

Section A

Program in Mat Lab the Timoshenko 2 Nodes Beam element with **reduce integration for the shear stiffness matrix**.

$$\mathbf{K}_b^{(e)} = \left(\frac{EI}{l}\right)^{(e)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad \text{(The point interpolation is exact for } \mathbf{K}_b^{(e)}\text{)}$$

$$\mathbf{K}_s^{(e)} = \left(\frac{GA^*}{l}\right)^{(e)} \begin{bmatrix} 1 & \frac{l^{(e)}}{2} & -1 & \frac{l^{(e)}}{2} \\ \cdots & \frac{(l^{(e)})^2}{4} & -\frac{l^{(e)}}{2} & \frac{(l^{(e)})^2}{4} \\ & \cdots & 1 & -\frac{l^{(e)}}{2} \\ \text{Simetr.} & & \cdots & \frac{(l^{(e)})^2}{4} \end{bmatrix} \quad \text{(Reduced integration)}$$

where:

$$K_b \; bending \; stiffness \; matrix$$
$$K_s \; shear \; stiffness \; matrix$$

Hint: For stress evaluation make gaus1 = gaus2 = 0.0

Reduced integration implies to decrease the shear effects in the stiffness matrix. Ks is modified to:

```
K_s = [   1    ,    len/2 ,    -1    ,    len/2 ;
        len/2  , len^2/4  , -len/2  , len^2/4 ;
         -1    ,  -len/2  ,     1    ,  -len/2 ;
        len/2  , len^2/4  , -len/2  ,  len^2/4 ];

K_s = K_s * const;
```

$$K^{(e)} = K_b^{(e)} + K_s^{(e)}$$

The stress is computed by using one numerical integration point instead of two gauss points. Given the code from Beam_Timoshenko_v1_2 folder, subroutine *Stress_Beam_Timoshenko.m* is edited as follow:

```
% One gauss point for stress evaluation
gaus0 =  0.0;     % One Gauss point for stresses evaluation

bmat_b=[ 0, -1/len,  0,  1/len];

bmat_s=[-1/len,-(1-gaus0)/2, 1/len,-(1+gaus0)/2];

Str1_g0 = D_matb*(bmat_b *transpose(u_elem));
Str2_g0 = D_mats*(bmat_s*transpose(u_elem));

Strnod(lnods(1),1)  = Strnod(lnods(1),1)+Str1_g0;
```

```
  Strnod(lnods(2),1) = Strnod(lnods(2),1)+Str1_g0;
  Strnod(lnods(1),2) = Strnod(lnods(1),2)+Str2_g0;
  Strnod(lnods(2),2) = Strnod(lnods(2),2)+Str2_g0;
  Strnod(lnods(1),3) = Strnod(lnods(1),3)+1;
  Strnod(lnods(2),3) = Strnod(lnods(2),3)+1;

end

for i = 1 : npnod
  Strnod(i,1) = Strnod(i,1)/Strnod(i,3);
  Strnod(i,2) = Strnod(i,2)/Strnod(i,3);
end
```

Reduced integration is used to avoid the shear locking effect. Beam problems are mainly affected by the bending moment. However, shear moment is a numerical conditioner numerically speaking. This is the shear locking effect. Also, when using Beam Timoshenko Theory for thin and thick beams, dimensions could significantly increase the shear moment contribution. This it because of the following equation:

$$\left(\frac{l^2}{3} \bar{K}_b + \frac{l^2 GA^*}{3EI} \bar{K}_s \right) a = \frac{l^3}{3EI} f = \bar{f}$$

Inertial moment (I) is dividing Ks.

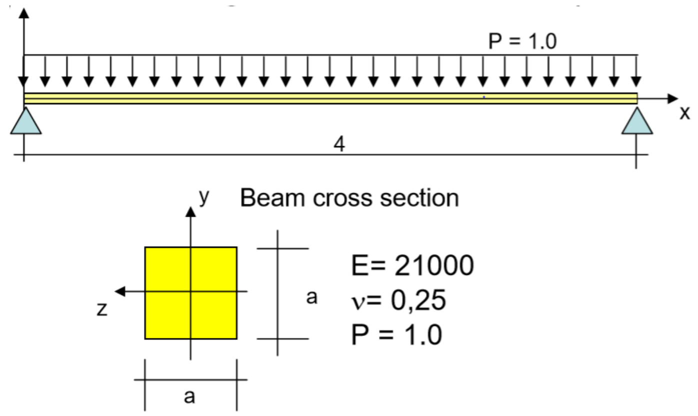
Reduced integration is used to reduce this shear locking effect.

Section B

Solve the following problem with a 64-element mesh with the:

- 2 nodes Euler Bernoulli element
- 2 nodes Timoshenko Full Integrate element
- 2 nodes Timoshenko Reduce Integration element

Compare maximum displacements, moments and shear for the 3 elements against the a/L relationship.



a values							
0.001	0.005	0.010	0.020	0.050	0.100	0.200	0.400

Mesh (Figure 1) was defined in GiD and then transformed as a problem description data (Table 1 and Figure 2 and 3) format that Matlab code for Euler-Bernoulli and Timoshenko codes could read.

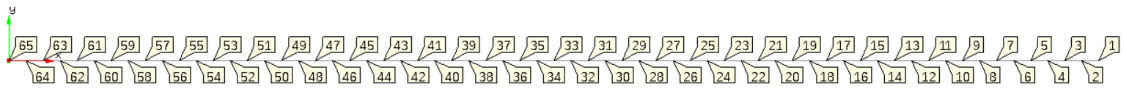


Figure 1. Mesh definition

It was created a problem description data for each a_i (beam side).

a (m)	Area cross section	L (m)	$I = \frac{L * a^3}{12}$
0,001	0,000001	4	3,3333E-10
0,005	0,000025	4	4,1667E-08
0,01	0,0001	4	3,3333E-07
0,02	0,0004	4	2,6667E-06
0,05	0,0025	4	4,1667E-05
0,1	0,01	4	0,00033333
0,2	0,04	4	0,00266667
0,4	0,16	4	0,02133333

Table 1. Problem data to use

```
%
%  Material Properties
%
  young  =   21000 ;%it i
  poiss  =  0.25;
  denss  =  1;
  a      =   0.001;  % La
  area   =   a^2;
  inertia=   3.33e-10;  %
  P=1; %Uniform distribut
%
% Coordinates
%
global coordinates
coordinates = [
4.00000 ;
3.93750 ;
3.87500 ;
3.81250 ;
```

Figure 2. Script extraction for problem description data. A=0.001 in this case

The aim of the exercise is to review the modelling behaviour for Euler-Bernoulli and Timoshenko theories as well as to evaluate its behaviour when beam thickness changes (for the same length). To that, full (Figure 3.a) and reduced (Figure 3.b) integration for Timoshenko theories had to be studied.

```
K_s = [   1    ,    len/2 ,    -1    ,    len/2 ;
        len/2 , len^2/3 , -len/2 , len^2/6 ;
         -1    ,  -len/2 ,     1    ,  -len/2 ;
        len/2 , len^2/6 , -len/2 ,  len^2/3 ];

K_s = K_s * const;
```

```
K_s = [   1    ,    len/2 ,    -1    ,    len/2 ;
        len/2 , len^2/4 , -len/2 , len^2/4 ;
         -1    ,  -len/2 ,     1    ,  -len/2 ;
        len/2 , len^2/4 , -len/2 ,  len^2/4 ];

K_s = K_s * const;
```

Figure 3. Timoshenko theory. Shear stiffness matrix for full (left) and reduced integration (right)

Beam Euler Bernoulli theory relies on the idea that shear stresses are not considered. So that, the problem is simplified to a line problem. This is for thin beams. On the other hand, Timoshenko is developed to compute thick beams because it considers the stresses. Reduced Timoshenko theory is used to compute both cases, thin and thick beams.

Three theories are evaluated for the given problem:

1. Euler-Bernoulli
2. Timoshenko full integration theory
3. Timoshenko reduced integration theory

Given cases can be studied as beam problems because a/L<=0.

a	a/L	Euler Bernouilli	Timoshenko Full integration	Timoshenko Reduced integration
0,001	0,00025	-3,9908E-05	-3,5966E-01	-4,7677E-01
0,005	0,00125	-3,9908E-06	-8,9669E-02	-9,5488E-02
0,01	0,0025	-3,9908E-06	-4,6430E-02	-4,7934E-02
0,02	0,005	-1,9939E-06	-2,3711E-02	-2,4092E-02
0,05	0,0125	-7,1963E-07	-8,8277E-03	-8,8779E-03
0,1	0,025	-3,9872E-07	-5,0308E-03	-5,0462E-03
0,2	0,05	-1,9934E-07	-2,6619E-03	-2,6657E-03
0,4	0,1	-9,9669E-08	-1,4748E-03	-1,4747E-03

Table 2. Maximum displacements for theories

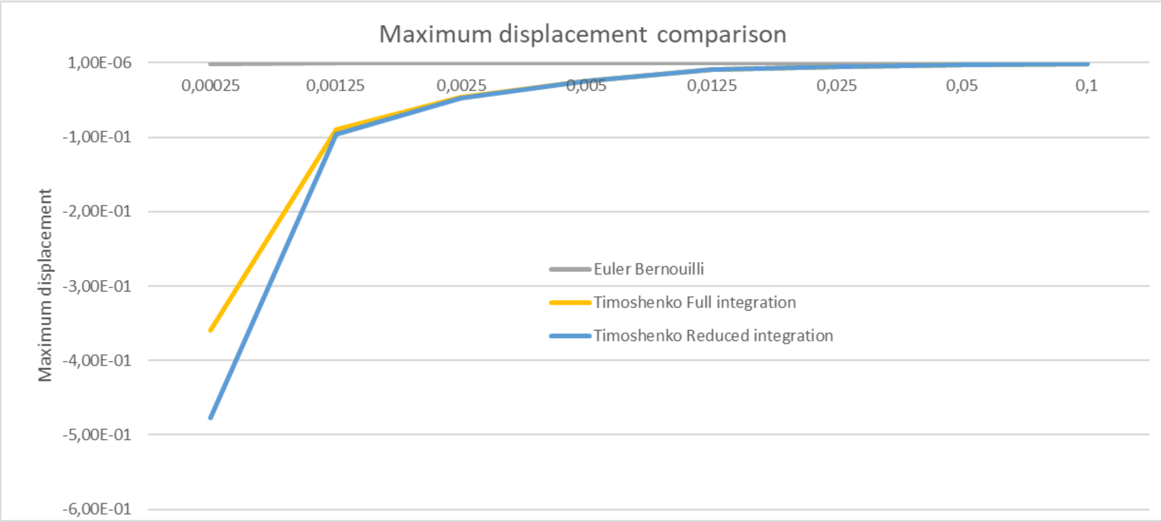


Figure 4. Maximum displacement evaluation

a	a/L	Euler Bernouilli	Timoshenko Full integration	Timoshenko Reduced integration
0,001	0,00025	2,63E-10	1,5077E-06	1,9990E-06
0,005	0,00125	5,79E-20	4,6921E-05	4,9976E-05
0,01	0,0025	2,63E-08	1,9359E-04	1,9990E-04
0,02	0,005	1,05E-07	7,8680E-04	7,9961E-04
0,05	0,0125	6,56E-07	4,9684E-03	4,9976E-03
0,1	0,025	2,63E-06	1,9925E-02	1,9990E-02
0,2	0,05	1,05E-05	7,9831E-02	7,9961E-02
0,4	0,1	4,20E-05	3,1958E-01	3,1984E-01

Table 3. Maximum moment for the problem

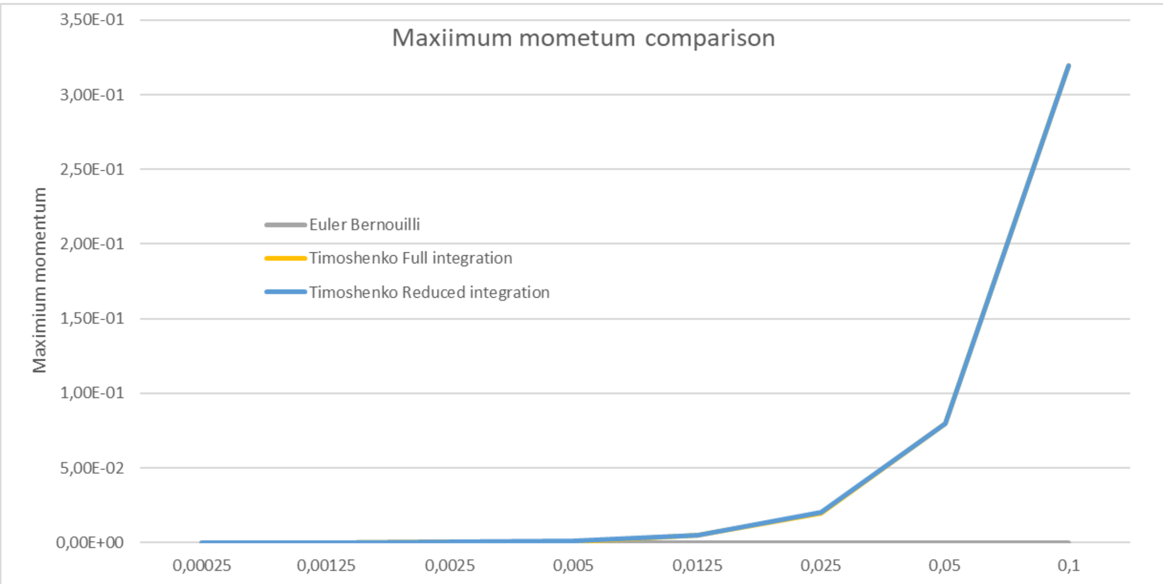


Figure 5. Maximum momentum evaluation

a	a/L	Timoshenko Full integration	Timoshenko Reduced integration
0,001	0,00025	1,1307E-06	1,9687E-06
0,005	0,00125	4,4008E-05	4,9219E-05
0,01	0,0025	1,8611E-04	1,9669E-04
0,02	0,005	7,6565E-04	7,8750E-04
0,05	0,0125	4,8721E-03	4,9219E-03
0,1	0,025	1,9577E-02	1,9687E-02
0,2	0,05	7,8528E-02	7,8755E-02
0,4	0,1	3,1456E-01	3,1500E-01

Table 4. Maximum shear stresses for the problem

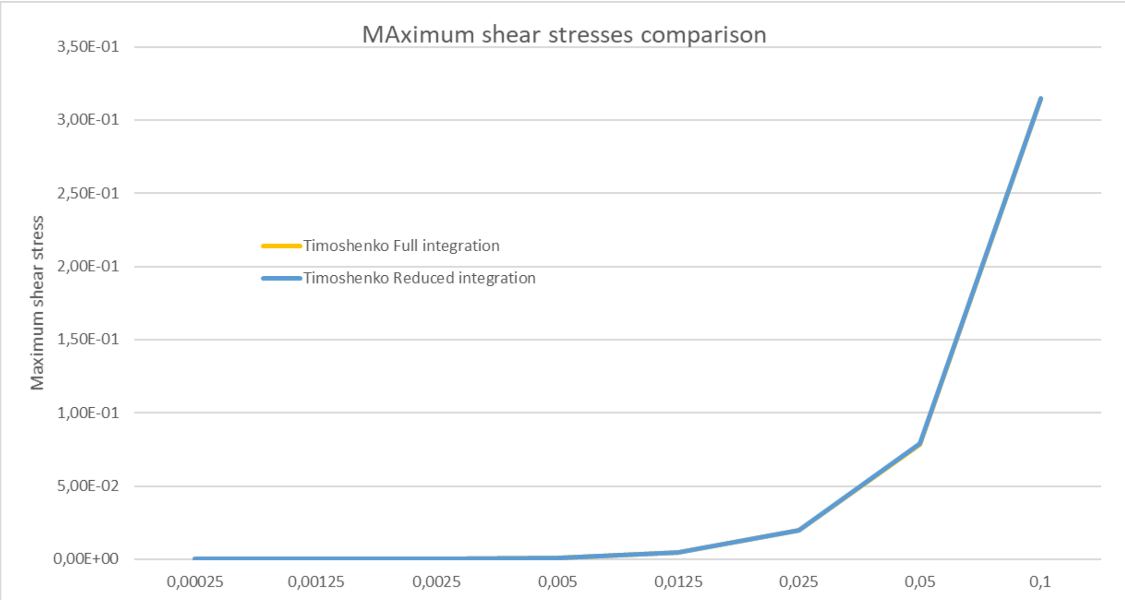
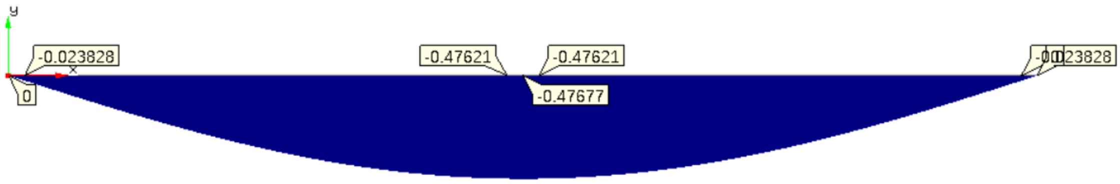


Figure 6. Maximum shear stressees evaluation

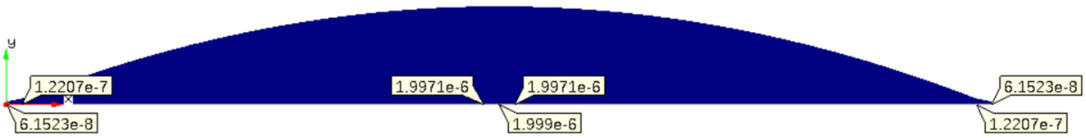
The thinner the beam, the bigger the magnitude values from the maximum displacement difference. As a value increases, it is noticed that the three methods have a similar behaviour. Reduced Timoshenko should be able to better approach to Euler Bernouilli results when the beam is thin and to Full Timoshenko when the beam is thicker.

Following graphics show how each theory performs for 0.001 and 0.4 m beam thickness.

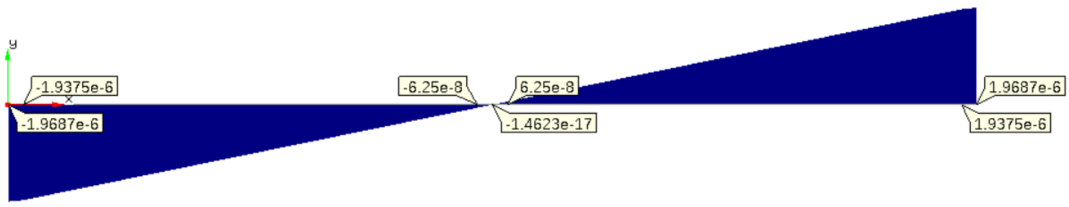
Timoshenko results – in this case for reduced integration



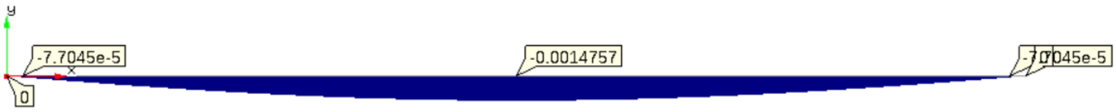
0.001 m. Displacement. Scalar line diagram view



0.001 m. Mz. Scalar line diagram view

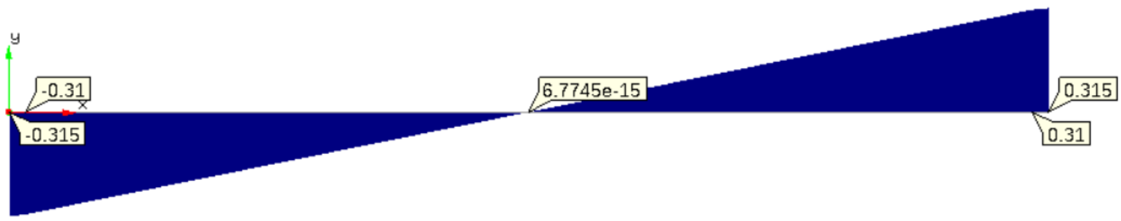


0.001 m. Shear force Qy. Scalar line diagram view

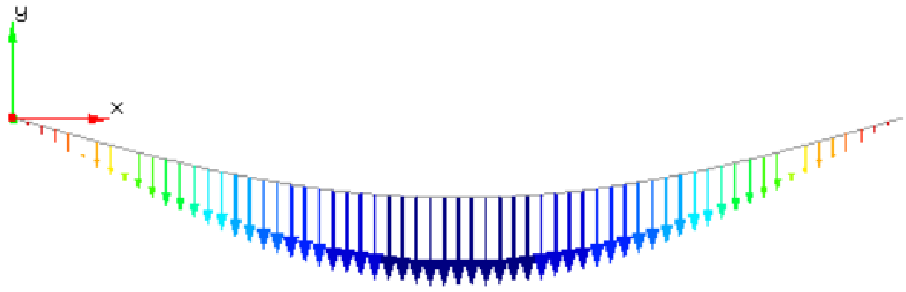


0.4 m. Displacement. Scalar line diagram view

0.4 m. Mz. Scalar line diagram view

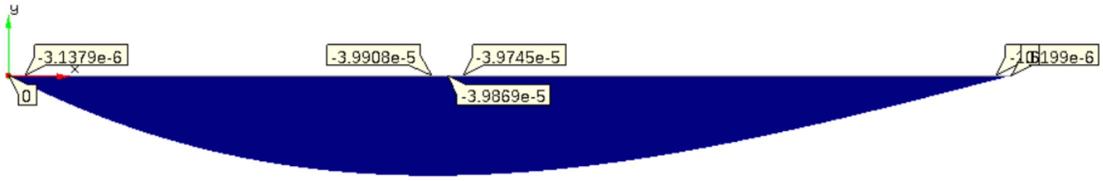


0.4 m. Shear force Qy. Scalar line diagram view



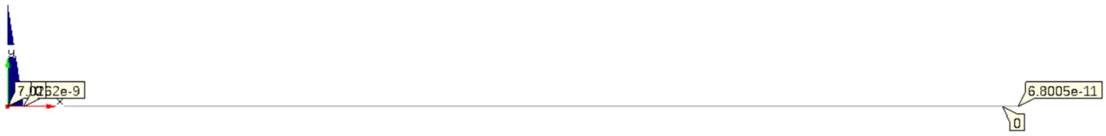
Beam deformation. Scale factor 1. Display vector view for displacements. t=0.001

Euler Bernoulli results (no shear stresses)



0.001 m. Displacement. Scalar line diagram view

0.001 m. Mz. Scalar line diagram view



0.001 m. y-reaction force (no shear stress). Scalar line diagram view