Assignment-5
Comp. Structural Mechanics & Dynamics

Prakhar Rastogi
Computational Mechanics

Assignment 5.1

$$\begin{bmatrix}1\\ \bar{x}\\ \bar{u}\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3\\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3\end{bmatrix}\begin{bmatrix}N_1^e(\xi)\\ N_2^e(\xi)\\ N_3^e(\xi)\end{bmatrix}$$

$\xi = -1$ $\quad$ $\xi_3 = 0$ $\quad$ $\xi_2 = +1$

$\bar{x}_1 = 0$ $\quad$ $\bar{x}_3 = \frac{L}{2}(1+2\alpha)$ $\quad$ $\bar{x}_2 = L$

$\xi_1 = -1 \qquad \xi_2 = +1 \qquad \xi_3 = 0$

$N_1 = \frac{\xi}{2}(\xi - 1) \qquad N_2 = \frac{\xi}{2}(\xi+1) \qquad N_3 = (1-\xi^2)$

$\frac{dN_1}{d\xi} = \frac{1}{2}(2\xi - 1) \qquad \frac{dN_2}{d\xi} = \frac{1}{2}(2\xi+1) \qquad \frac{dN_3}{d\xi} = -2\xi$

$$J = \frac{d\bar{x}}{d\xi} = \frac{dN_1}{d\xi}\bar{x}_1 + \frac{dN_2}{d\xi}\bar{x}_2 + \frac{dN_3}{d\xi}\bar{x}_3$$

$$= \frac{1}{2}(2\xi-1)(0) + \frac{1}{2}(2\xi+1)(L) + (-2\xi)\left(\frac{L}{2}\right)(1+2\alpha)$$

$$= \frac{L}{2}(2\xi+1-2\xi-4\xi\alpha) = \frac{L}{2}(1-4\alpha\xi)$$

$$\boxed{J = \frac{L}{2}(1-4\alpha\xi)}$$

when $\xi = -1$, $J = \frac{L}{2}(1+4\alpha)$

$$J = 0 \Rightarrow \frac{L}{2}(1+4\alpha) = 0 \Rightarrow \alpha = -\frac{1}{4}$$

$\xi = +1$, $J = \frac{L}{2}(1-4\alpha\xi)$

$$J = 0 \Rightarrow \frac{L}{2}(1-4\alpha) = 0 \Rightarrow \alpha = \frac{1}{4}$$

$\epsilon = \frac{du}{dx} = Bu^e$; $B \rightarrow$ Strain Matrix

$$B = \frac{dN}{dx} = J^{-1}\frac{dN}{d\xi} = \left(\frac{d\xi}{dx}\right)\frac{dN}{dx}$$

$$\therefore B = \frac{2}{L(1-4\xi\alpha)}\begin{bmatrix}\frac{(2\xi-1)}{2} & \frac{(2\xi+1)}{2} & -2\xi\end{bmatrix}$$

1 and 2 are the end pts of the element

At point 1 ; $\xi = -1$ and $\alpha = \frac{1}{4}$ $\Rightarrow L(1-4\xi\alpha) = 0$

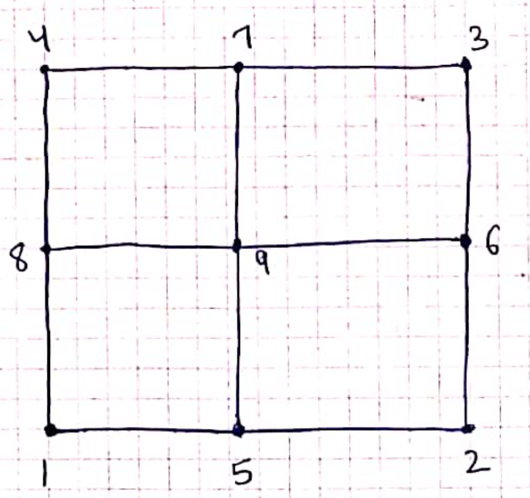
$\therefore$ B becomes infinity

Similarly, at point 2 $\xi = +1$ and $\alpha = (-1/4)$

$\Rightarrow L(1-4\alpha\xi) = 0$ $\quad\therefore$ B becomes infinity.

It can be concluded that at end pts. 1 and 2, strain attains Singularity.

Assignment 5.2



Node	ξ_i	η_i
1	-1	-1
2	1	-1
3	1	1
4	-1	1
5	0	-1
6	1	0
7	0	1
8	-1	0
9	0	0

Nodes(i)

1,2,3,4 $\quad N_i = \frac{1}{4}(\xi^2+\xi\xi_i)(\eta^2+\eta\eta_i)$

5,6,7,8 $\quad N_i = \frac{1}{2}\eta_i^2(\eta^2+\eta\eta_i)(1-\xi^2) + \frac{1}{2}\xi_i^2(\xi^2+\xi\xi_i)(1-\eta^2)$

9 $\quad N_9 = (1-\xi^2)(1-\eta^2)$

$$N_1 = \frac{1}{4}(\xi-1)(\eta-1)\xi\eta$$

$$N_2 = \frac{1}{4}(\xi+1)(\eta-1)\xi\eta$$

$$N_3 = \frac{1}{4}(\xi+1)(\eta+1)\xi\eta$$

$$N_4 = \frac{1}{4}(\xi-1)(\eta+1)\xi\eta$$

$$N_5 = \frac{1}{2}(\eta\mp1)(1-\xi^2)\eta$$

$$N_6 = \frac{1}{2}(\xi+1)(1-\eta^2)\xi$$

$$N_7 = \frac{1}{2}(\eta+1)(1-\xi^2)\eta$$

$$N_8 = \frac{1}{2}(\xi\mp1)(1-\eta^2)\xi$$

$$N_9 = (1-\xi^2)(1-\eta^2)$$

Node	x	y
1	0	0
2	L	0
3	L	L
4	0	L
5	$\frac{L}{2}(2\alpha+1)$	0
6	L	$\frac{L}{2}$
7	$\frac{L}{2}$	L
8	0	$\frac{L}{2}$
9	$\frac{L}{2}$	$\frac{L}{2}$

$$\frac{\partial N_1}{\partial \xi} = \frac{1}{4}\eta(\eta-1)(2\xi-1) \qquad \frac{\partial N_1}{\partial \eta} = \frac{1}{4}\xi(\xi-1)(2\eta-1)$$

$$\frac{\partial N_2}{\partial \xi} = \frac{1}{4}\eta(\eta-1)(2\xi+1) \qquad \frac{\partial N_2}{\partial \eta} = \frac{1}{4}\xi(\xi+1)(2\eta-1)$$

$$\frac{\partial N_3}{\partial \xi} = \frac{1}{4}\eta(\eta+1)(2\xi+1) \qquad \frac{\partial N_3}{\partial \eta} = \frac{1}{4}\xi(\xi+1)(2\eta+1)$$

$$\frac{\partial N_4}{\partial \xi} = \frac{1}{4}\eta(\eta+1)(2\xi-1) \qquad \frac{\partial N_4}{\partial \eta} = \frac{1}{4}\xi(\xi-1)(2\eta+1)$$

$$\frac{\partial N_5}{\partial \xi} = \frac{1}{2}\eta(\eta-1)(-2\xi) = -\eta\xi(\eta-1) \qquad \frac{\partial N_5}{\partial \eta} = \frac{1}{2}(1-\xi^2)(2\eta-1)$$

$$\frac{\partial N_6}{\partial \xi} = \frac{1}{2}(1-\eta^2)(2\xi+1) \qquad \frac{\partial N_6}{\partial \eta} = \frac{1}{2}\xi(\xi+1)(-2\eta) = -\eta\xi(\xi+1)$$

$$\frac{\partial N_7}{\partial \xi} = \frac{1}{2}\eta(\eta+1)(-2\xi) = -\eta\xi(\eta+1) \qquad \frac{\partial N_7}{\partial \eta} = \frac{1}{2}(1-\xi^2)(2\eta+1)$$

$$\frac{\partial N_8}{\partial \xi} = \frac{1}{2}(1-\eta^2)(2\xi-1) \qquad \frac{\partial N_8}{\partial \eta} = \frac{1}{2}\xi(\xi+1)(-2\eta) = -\eta\xi(\xi-1)$$

$$\frac{\partial N_9}{\partial \xi} = -2\xi(1-\eta^2) \qquad \frac{\partial N_9}{\partial \eta} = -2\eta(1-\xi^2)$$

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial \bar{x}}{\partial \xi} & \frac{\partial \bar{y}}{\partial \xi} \\ \frac{\partial \bar{x}}{\partial \eta} & \frac{\partial \bar{y}}{\partial \eta} \end{bmatrix}$$

For pt-2 $(\xi, \eta) = (1, -1)$

$$J_{11} = \frac{\partial \bar{x}}{\partial \xi} = \frac{\partial N_2}{\partial \xi}(L) + \frac{\partial N_3}{\partial \xi}(L) + \frac{\partial N_5}{\partial \xi}\left(\frac{L}{2} + \alpha L\right) + \frac{\partial N_6}{\partial \xi}(L) + \frac{\partial N_7}{\partial \xi}\left(\frac{L}{2}\right) + \frac{\partial N_9}{\partial \xi}\left(\frac{L}{2}\right)$$
$$= \frac{L}{2} - 2\alpha L$$

$$J_{12} = \frac{\partial \bar{y}}{\partial \xi} = \frac{\partial N_3}{\partial \xi}(L) + \frac{\partial N_4}{\partial \xi}(L) + \frac{\partial N_6}{\partial \xi}\left(\frac{L}{2}\right) + \frac{\partial N_7}{\partial \xi}(L) + \frac{\partial N_8}{\partial \xi}\left(\frac{L}{2}\right) + \frac{\partial N_9}{\partial \xi}\left(\frac{L}{2}\right) = 0$$

$$J_{21} = \frac{\partial \bar{x}}{\partial \eta} = \frac{\partial N_2}{\partial \eta}(L) + \frac{\partial N_3}{\partial \eta}(L) + \frac{\partial N_5}{\partial \eta}\left(\frac{L}{2} + \alpha L\right) + \frac{\partial N_6}{\partial \eta}(L) + \frac{\partial N_7}{\partial \eta}\left(\frac{L}{2}\right) + \frac{\partial N_9}{\partial \eta}\left(\frac{L}{2}\right)$$
$$= -\cancel{\frac{3L}{2}} + \cancel{\frac{3L}{2}} = 0$$

$$J_{22} = \frac{\partial \bar{y}}{\partial \eta} = \frac{\partial N_3}{\partial \eta}(L) + \frac{\partial N_4}{\partial \eta}(L) + \frac{\partial N_6}{\partial \eta}\left(\frac{L}{2}\right) + \frac{\partial N_7}{\partial \eta}(L) + \frac{\partial N_8}{\partial \eta}\left(\frac{L}{2}\right) + \frac{\partial N_9}{\partial \eta}\left(\frac{L}{2}\right)$$
$$= \frac{3L}{2} - L = \frac{L}{2}$$

$$\det(J) = |J| = \left(\frac{L}{2}\right)^2(1 - 4\alpha) = 0 \Rightarrow \boxed{\alpha = \frac{1}{4}}$$

Coordinates of Pt-5 $\left(\frac{L}{2}(1 + 2\alpha), 0\right)$

Determinant of Jacobian vanishes at $\alpha = \frac{1}{4}$

$\therefore$ Coordinates of 5 pt should be $\left(\frac{3L}{4}, 0\right)$ so that the Jacobian is zero.