ASSIGNMENT 5
"Convergence Requirements"

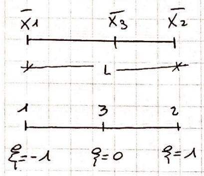
Assignment 5.1

Isoparametric definition of the straight-node bar element (local system)

$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix}$$

isoparametric coordinates $\xi = \begin{cases} -1 \\ 0 \\ 1 \end{cases}$

N_1^e, N_2^e, N_3^e are the shape functions.



CARTESIAN COORDINATES

$\bar{x}_1 = 0$
$\bar{x}_2 = L$
$\bar{x}_3 = \frac{1}{2}L + \alpha L$

NATURAL COORDINATES (in quadratic element)

We know that the relation between the coordinate of the nodes and the cartesian coordinates is:

$$\bar{x} = N_1 \bar{x}_1 + N_2 \bar{x}_2 + N_3 \bar{x}_3$$

We can write the ~~coordinate~~ shape functions:

$$\begin{cases} N_1 = \xi(\xi-1)/2 \\ N_2 = \xi(\xi+1)/2 \\ N_3 = -(\xi-1)(\xi+1) = 1-\xi^2 \end{cases}$$

Now we can introduce in the $\bar{x}$ definition the 3 shape functions and the cartesian coordinates values $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$:

$$\bar{x} = \left[\frac{1}{2}\xi(\xi-1)\right]\cdot 0 + \left[\frac{1}{2}\xi(1+\xi)\right]\cdot L + \left[1-\xi^2\right]\left(\frac{1}{2}L + \alpha L\right) =$$

$$= \frac{1}{2}(1+\xi)(2\alpha - 2\xi\alpha + 1)$$

The Jacobian can be defined as $J = \frac{dx}{d\xi}$, so:

$$J = \frac{dx}{d\xi} = \left(\frac{1}{2} + \frac{1}{2}(2\xi)\right)L + \left(\frac{-2\xi}{2}L\right) - 2\xi\alpha L = \frac{L}{2} - 2\xi\alpha L$$

The Jacobian is positive if $-\frac{1}{4} < \alpha < \frac{1}{4}$, in fact if we apply the condition below:

$$J = 0 \Rightarrow \frac{L}{2} - 2\xi\alpha L = 0 \rightarrow \alpha = \frac{1}{4\xi}$$

and we know that $\xi = \begin{cases} -1 \\ 0 \\ 1 \end{cases}$ so if $\begin{cases} \xi = -1 \rightarrow \alpha = -1/4 \\ \xi = 0 \rightarrow \alpha = 0 \\ \xi = 1 \rightarrow \alpha = 1/4 \end{cases}$

So the minimu α for which $J = 0$ are $\pm \frac{1}{4}$.

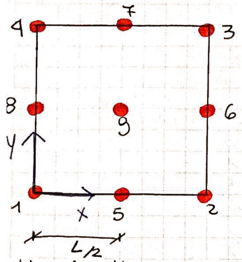
Strain desplacement matrix:

$$B = J^{-1} \frac{dN}{d\xi} = \frac{2}{1 - 4\alpha\xi} \left[\xi - \frac{1}{2} ; \ \xi + \frac{1}{2} ; \ -2\xi \right]$$

* for $\xi = 1 \rightarrow \lim_{\alpha \to \frac{1}{4}} B = [\infty ; \infty ; \infty]$

$\xi = -1 \rightarrow \lim_{\alpha \to -\frac{1}{4}} B = [\infty ; \infty ; \infty]$

Assignment 5.2.

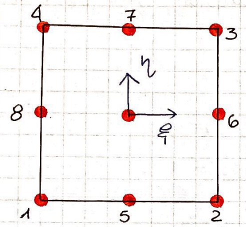


We have 9-node quadrilateral element.

node 1 $\begin{cases} x_i = 0 \\ y_i = 0 \\ t = 1 \end{cases}$

the Jacobian can be define as $\rightarrow$ $|J| = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix}$

We fix the natural coordinates in the node number 9.



We can describe the shape functions as below:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)(\xi\eta)$$

$$N_2 = -\frac{1}{4}(1+\xi)(1-\eta)\xi\eta$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)\xi\eta$$

$$N_4 = -\frac{1}{4}(1-\xi)(1+\eta)\xi\eta$$

$$N_5 = -\frac{1}{2}(1-\xi^2)(1-\eta)\eta$$

$$N_6 = \frac{1}{2}(1+\xi)(1-\eta^2)\xi$$

$$N_7 = \frac{1}{2}(1-\xi^2)(1+\eta)\eta$$

$$N_8 = -\frac{1}{2}(1-\xi)(1-\eta^2)\xi$$

$$N_9 = (1-\xi^2)(1-\eta^2)$$

where $-1 \leq \xi \leq 1$ and $-1 \leq \eta \leq 1$

We can define the coordinates of all 9 node:

NODE	X_i	Y_i
1	0	0
2	L	0
3	L	L
4	0	L
5	$\frac{L}{2}+\alpha L$	0
6	L	L/2
7	L/2	L
8	0	L/2
9	L/2	L/2

natural coordinates

Cartesian coordinates →

NODE	ξ	η
1	-1	-1
2	0	-1
3	1	-1
4	1	0
5	1	1
6	0	1
7	-1	1
8	-1	0
9	0	0

$$x = \sum_{i=1}^{9} N_i(\xi;\eta)\, x_i$$

$$y = \sum_{i=1}^{9} N_i(\xi,\eta)\, y_i$$

As we said before the Jacobian is defined as $J = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix}$

So we will have:

$$J_{11}(\xi,\eta) = -\frac{1}{4}(2\xi-1)(\eta-1)\eta + \frac{1}{4}(2\xi+1)(\eta-1)\eta + $$
$$+\frac{1}{4}(2\xi+1)(\eta+1)\eta - \frac{1}{4}(2\xi-1)(\eta+1)\eta - \alpha(\eta-1)\xi\eta - \frac{1}{2}(2\xi+1)(\eta+1)(\eta-1)$$
$$+\frac{1}{2}(2\xi-1)(\eta+1)(\eta-1).$$

$$J_{12}(\xi,\eta) = -\frac{1}{4}(2\xi-1)(\eta-1)\eta - \frac{1}{4}(2\xi+1)(\eta-1)\eta + \frac{1}{4}(2\xi+1)(\eta+1)\eta + \frac{1}{4}(2\xi-1)(\eta+1$$
$$+(\eta-1)\xi\eta - (\eta+1)\xi\eta.$$

$$J_{21}(\xi,\eta) = -\frac{1}{4}(\xi-1)(2\eta-1)\xi + \frac{1}{4}(\xi+1)(2\eta-1)\xi + \frac{1}{4}(\xi+1)(2\eta+1)\xi - \frac{1}{4}(\xi-1)(2\eta+1$$
$$-\frac{\alpha}{2}(2\eta-1)(\xi+1)(\xi-1) - (\xi+1)\xi\eta + (\xi-1)\xi\eta.$$

$$J_{22}(\xi,\eta) = -\frac{1}{4}(\xi-1)(2\eta-1)\xi - \frac{1}{4}(\xi+1)(2\eta-1)\xi + \frac{1}{4}(\xi+1)(2\eta+1)\xi + \frac{1}{4}(\xi-1)(2\eta+1$$
$$+\frac{1}{2}(2\eta-1)(\xi-1)(\xi+1) - \frac{1}{2}(2\eta+1)(\xi-1)(\xi+1).$$

We can see that for node-2 the Jacobian will be vanish:

$\det(J) = 0$ in node-2 = $(-1;1)$ $\longrightarrow$ $\alpha = \frac{1}{4}$

Therefore we can calculate the coordinates of node-5:

$$X_5 = \left[\frac{L}{2} + \alpha L \,;\, 0\right] = \left[\frac{3L}{4} \,;\, 0\right]$$