Computational Structural Mechanics and Dynamics

Masters in Numerical Methods

Assignment 5

Convergence Requirements

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1 Assignment 5.1

1.1 Part A

Here we have to compute the Jacobian of the element. We define

$$x_1 = 0 \qquad x_2 = l \qquad x_2 = (1/2 + \alpha)l \tag{1}$$

The shape functions for a three noded linear element are given by

$$N_1^e = \frac{\xi}{2}(1-\xi) \qquad N_2^e = \frac{\xi}{2}(1+\xi) \qquad N_2^e = 1-\xi^2 \tag{2}$$

The x coordinate of a point within the element is written in the iso-parametric formulation as

$$x = \sum N_i(\xi)x_i \tag{3}$$

Substituting the values of the shape functions and the values of the x_i :

$$x = \frac{\xi}{2}(1-\xi)(0) + \frac{\xi}{2}(1+\xi)l + 1 - \xi^2((\frac{1}{2}+\alpha)l \tag{4}$$

$$x = -\frac{l}{2}(2\xi^2\alpha - \xi - 2\alpha - 1) \tag{5}$$

The Jacobian is defined as :

$$J = \frac{\partial x}{\partial \xi} = \frac{l}{2}[1 - 4\xi\alpha] \tag{6}$$

The Jacobian is positive for $\frac{-1}{4} \leq \alpha \leq \frac{1}{4}$, thus we can see that for the extreme values of ξ the value of the Jacobian is zero for $\alpha = \frac{-1}{4}$ corresponding to $\xi = -1$ and $\alpha = \frac{1}{4}$ for $\xi = 1$.

1.2 Part B

The strain matrix **B** is defined as

$$\mathbf{B} = \frac{\partial \xi}{\partial x}[\frac{dN_1}{d\xi}, \frac{dN_2}{d\xi}, \frac{dN_3}{d\xi}] \tag{7}$$

In this case, it is given by

$$\mathbf{B} = \frac{2}{(1-4\xi\alpha)l^e}[\xi - \frac{1}{2}, -2\xi, \xi + \frac{1}{2}] \tag{8}$$

These equations imply that $\alpha = \frac{1}{4}$ and $\alpha = \frac{-1}{4}$ are the points of singularity for Strain as it tends to infinity for the extreme values of $\xi = 1$ and $\xi = -1$ respectively.

2 Assignment 5.2

For a nine noded biquadratic plane stress element the shape functions are as follows:

$$N_1 = \frac{1}{4}\xi\eta(\eta-1)(\xi-1) \qquad N_2 = \frac{1}{4}\xi\eta(\eta+1)(\xi-1)$$

$$N_3 = \frac{1}{4}\xi\eta(\eta-1)(\xi+1) \qquad N_4 = \frac{1}{4}\xi\eta(\eta+1)(\xi+1)$$

$$N_5 = \frac{1}{2}(1-\xi^2)\eta(\eta-1) \qquad N_6 = \frac{1}{2}(1-\eta^2)\xi(\xi+1)$$

$$N_7 = \frac{1}{2}(1-\xi^2)\eta(\eta+1) \qquad N_8 = \frac{1}{2}(1-\eta^2)\xi(\xi-1)$$

$$N_9 = (1-\xi^2)(1-\eta^2)$$

These shape functions have the value one at the corresponding node and zero at other nodes.

To fix the coordinate system, we take the node one to be the global origin. If l is the length of each side of the square we can take $x_5 = ((\frac{1}{2}+\alpha)l, 0)$. As per the question, we need to find the value of α for which the Jacobian is zero at node 2, which corresponds to $(\xi, \eta) = (1, -1)$.

The Jacobian is given as follows :

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \tag{9}$$

The derivatives and Jacobian are calculated using the matlab code. The coordinates of the points are as follows

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0.5+\alpha & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0.5 & 1 & 0.5 & 0.5 \end{bmatrix}^T l \tag{10}$$

The iso-parametric mapping is given by for $i = 1...9$

$$x = \sum N_i(\xi, \eta)x_i \qquad y = \sum N_i(\xi, \eta)y_i \tag{11}$$

After solving this problem with symbolic variables in MATLAB we find that

$$|J| = l^2(\frac{1}{4} - \alpha) = 0 \tag{12}$$

Therefore we find that $\alpha = \frac{1}{4}$

The value of $\alpha = \frac{1}{4}$ corresponds to $x_5 = (\frac{3l}{4}, 0)$ which lies on the side 1-2. This implies that the nodes located in a 2D quadrilateral element must remain between $(\frac{l}{2}, \frac{3l}{4})$ just like the 1D case seen in example 5.1.

3 Appendix

A MATLAB code for symbolic matrix multiplication written for this assignment.

```
%% Assignment 5 Question 2
clear all;
syms xi eta a l

N{1} = (xi-1)*(eta-1)*xi*eta/4;
N{2} = (xi+1)*(eta-1)*xi*eta/4;
N{3} = (xi+1)*(eta+1)*xi*eta/4;
N{4} = (xi-1)*(eta+1)*xi*eta/4;

N{5} = (xi+1)*(xi-1)*eta*(1 - eta)/2;
N{6} = (xi+1)*(eta+1)*xi*(1 - eta)/2;
N{7} = -(xi+1)*(xi-1)*eta*(1+eta)/2;
N{8} = -(xi-1)*xi*(eta-1)*(eta+1)/2;

N{9} = (1-xi^2)*(1-eta^2);

X = l*[0,0; 1,0;1,1;0,1;1/2 + a,0; 1, 1/2; 1/2,1 ; 0,1/2;
    1/2,1/2];

sumX = 0;
sumY = 0;
for i = 1:9;
    sumX = sumX + N{i}*X(i,1);
    sumY = sumY + N{i}*X(i,2);
end

J11 = diff(sumX,xi);
J12 = diff(sumY,xi);
J21 = diff(sumX,eta);
J22 = diff(sumY,eta);

J = [J11,J12;J21,J22];

detJ = det(J);

detJ = subs(detJ,xi,1);
detJ = subs(detJ,eta,-1);
anew = solve(detJ == 0,a);
disp(anew)
```