Universitat Politècnica de Catalunya

Escola Tècnica Superior d'Enginyers de Camins, Canals i Ports

MASTER EN INGENIERÍA ESTRUCTURAL Y DE LA CONSTRUCCIÓN

Asignatura:

COMPUTATIONAL STRUCTURAL MECHANICS AND DYNAMICS

Assignment 5

On "Isoparametric representation" and "Convergence rerquirements"

By

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Assignment 5.1:

On "Isoparametric representation":

Consider a three-node bar element referred to the natural coordinate ξ. The two end nodes and the mid node are identified as 1, 2 and 3 respectively. The natural coordinates of nodes 1, 2 and 3 are ξ = −1, ξ = 1 and ξ = 0, respectively. The variation of the shape functions N1(ξ), N2(ξ) and N3(ξ) is sketched in the figure below. These functions must be quadratic polynomials in ξ:

$$N_1^e(\xi) = a_0 + a_1\xi + a_2\xi^2$$

$$N_2^e(\xi) = b_0 + b_1\xi + b_2\xi^2$$

$$N_3^e(\xi) = c_0 + c_1\xi + c_2\xi^2$$

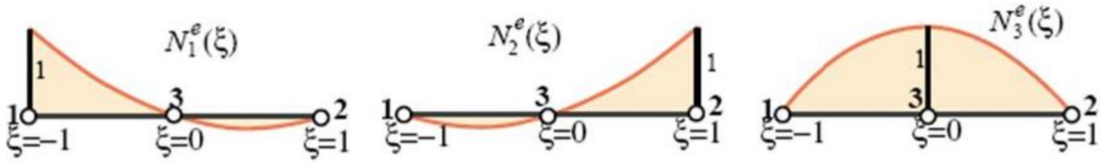

Figure.- Isoparametric shape functions for 3-node bar element (sketch).

a) Determine the coefficients a0,…,c2 using the node value conditions depicted in figure. For exemple $N_1^e(\xi) = 1$ for ξ=1 and 0 for the rest of natural coordinates. The rest of the nodes follow the same scheme.

b) Verify that their sum is identically one.

c) Calculate their derivatives respect to the natural coordinates.

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a) For $N_1^e(\xi)$, the system of equations results

$$\begin{cases} N_1^e(-1) = 1 \\ N_1^e(0) = 0 \\ N_1^e(1) = 0 \end{cases}$$

Operating

$$N_1^e(0) = a_0 + a_1\xi + a_2\xi^2 = a_0 + a_1 0 + a_2 0^2 = 0 => a_0 = 0$$

$$N_1^e(-1) = a_1\xi + a_2\xi^2 = -a_1 + a_2 = 1 => a_2 = a_1 + 1$$

$$N_1^e(1) = a_1\xi + a_2\xi^2 = a_1 + a_2 = a_1 + a_1 + 1 = 0 => 2a_1 = -1 => a_1 = -\frac{1}{2}\ ;\ a_2 = \frac{1}{2}$$

Results

$$N_1^e(\xi) = -\frac{1}{2}\xi + \frac{1}{2}\xi^2$$

For $N_2^e(\xi)$, the system of equations results

$$\begin{cases} N_2^e(-1) = 0 \\ N_2^e(0) = 0 \\ N_2^e(1) = 1 \end{cases}$$

$$N_2^e(0) = b_0 + b_1\xi + b_2\xi^2 = b_0 + b_1 0 + b_2 0^2 = 0 => b_0 = 0$$

$$N_2^e(-1) = b_1\xi + b_2\xi^2 = -b_1 + b_2 = 0 => b_2 = b_1$$

$$N_2^e(1) = b_1\xi + b_2\xi^2 = b_1 + b_2 = 2b_1 = 1 => b_2 = b_1 = \frac{1}{2}$$

$$N_2^e(\xi) = \frac{1}{2}\xi + \frac{1}{2}\xi^2$$

For $N_3^e(\xi)$, the system of equations results

$$\begin{cases} N_3^e(-1) = 0 \\ N_3^e(0) = 1 \\ N_3^e(1) = 0 \end{cases}$$

$$N_3^e(0) = c_0 + c_1\xi + c_2\xi^2 = c_0 + c_1 0 + c_2 0^2 = 1 => c_0 = 1$$

$$N_3^e(-1) = 1 + c_1\xi + c_2\xi^2 = 1 - c_1 + c_2 = 0 => c_1 = 1 + c_2$$

$$N_3^e(1) = 1 + c_1\xi + c_2\xi^2 = 1 + c_1 + c_2 = 1 + 1 + 2c_2 = 0 => c_2 = -1\,; c_1 = 0$$

$$N_3^e(\xi) = 1 - \xi^2$$

b) The shape functions must verify

$$\sum N^e = 1\ \forall\xi$$

$$N_1^e(\xi) + N_2^e(\xi) + N_3^e(\xi) = -\frac{1}{2}\xi + \frac{1}{2}\xi^2 + \frac{1}{2}\xi + \frac{1}{2}\xi^2 + 1 - \xi^2 = 1$$

c) Derivation respect ξ

$$\frac{\partial N_1^e}{\partial \xi} = \frac{\partial\left(-\frac{1}{2}\xi + \frac{1}{2}\xi^2\right)}{\partial \xi} = -\frac{1}{2} + \xi$$

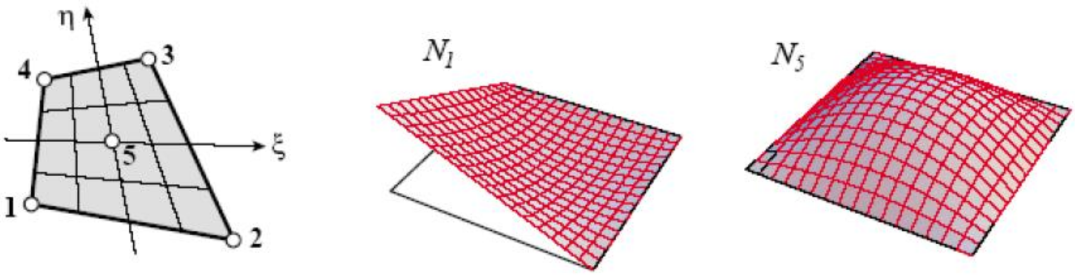
$$\frac{\partial N_2^e}{\partial \xi} = \frac{\partial\left(\frac{1}{2}\xi + \frac{1}{2}\xi^2\right)}{\partial \xi} = \frac{1}{2} + \xi$$

$$\frac{\partial N_3^e}{\partial \xi} = \frac{\partial(1 - \xi^2)}{\partial \xi} = -2\xi$$

Assignment 5.2:

On “Isoparametric representation”:

A five node quadrilateral element has the nodal configuration shown if the figure with two perspective views of $N_1^e$and N_5^e. Find five shape functions N_i^e, i=1,…,5 that satisfy compatibility and also verify that their sum is unity.



Hint: develop N_5(ξ,η) first for the 5-node quad using the line-product method. Then the corner shape functions Ni(ξ,η), i=1,2,3,4, for the 4-node quad (already given in the notes). Finally combine Ni = Ni + αN5 determining α so that all Ni vanish node 5. Check that N1 + N2 + N3 + N4 + N5 = 1 identically.

1-2 line have the equation

$$\eta = -1 \; \forall \, \xi$$

3-4 line have the equation

$$\eta = 1 \; \forall \, \xi$$

1-4 line have the equation

$$\xi = -1 \; \forall \, \eta$$

2-3 line have the equation

$$\xi = 1 \; \forall \, \eta$$

The $N_5(\xi,\eta)$ must satisfy

$$N_5(0{,}0) = 1$$

$$N_5(\forall \, \xi, \pm 1) = 0$$

$$N_5(\pm 1, \forall \, \eta) = 0$$

Using the line product method

$$N_5(\xi,\eta) = 1(1-\xi)(1+\xi)(1-\eta)(1+\eta) = 1(1-\xi^2)(1-\eta^2)$$

From slides for 4-node quad

$$N'^e_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N'^e_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N'^e_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N'^e_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

Adding in generic shape function i

$$N^e_i = N'^e_i + \alpha N^e_5 = \frac{1}{4}(1 \pm \xi)(1 \pm \eta) + \alpha(1-\xi^2)(1-\eta^2)$$

Must satisfy

$$N^e_i(0,0) = 0 \rightarrow \frac{1}{4}(1 \pm 0)(1 \pm 0) + \alpha(1-0^2)(1-0^2) = \frac{1}{4} + \alpha = 0$$

$$\alpha = -\frac{1}{4}$$

So

$$N^e_1 = \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2)$$

$$N^e_2 = \frac{1}{4}(1+\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2)$$

$$N^e_3 = \frac{1}{4}(1+\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2)$$

$$N^e_4 = \frac{1}{4}(1-\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2)$$

Adding all shape functions

$$N_1^e + N_2^e + N_3^e + N_4^e + N_5^e =$$

$$= \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2) + \frac{1}{4}(1+\xi)(1-\eta)$$
$$- \frac{1}{4}(1-\xi^2)(1-\eta^2) + \frac{1}{4}(1+\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2)$$
$$+ \frac{1}{4}(1-\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2) + (1-\xi^2)(1-\eta^2) =$$

$$= \frac{1}{4}(1-\xi)(1-\eta) + \frac{1}{4}(1+\xi)(1-\eta) + \frac{1}{4}(1+\xi)(1+\eta) + \frac{1}{4}(1-\xi)(1+\eta) =$$

$$= \frac{1}{4}(1-\eta+\xi\eta-\xi+1-\eta+\xi-\xi\eta+1+\eta+\xi\eta+\xi+1+\eta-\xi-\xi\eta) = \frac{4}{4} = 1$$

Assignment 5.3:

On “Convergence requirements”:

Which minimum integration rules of Gauss-product type gives a rank sufficient stiffness matrix for these elements:

1. the 8-node hexahedron
2. the 20-node hexahedron
3. the 27-node hexahedron
4. the 64-node hexahedron

For each

$$n_f = 3n, n_R = 6, n_E = 6$$

So

$$n_G \geq \frac{n_f - n_R}{n_E} = \frac{3n-6}{6}$$

Element	n	nF	nF - nR	Min nG
8 - node hexahedron	8	24	18	3
20 - node hexahedron	20	60	54	9
27 - node hexahedron	27	81	75	13
64 - node hexahedron	64	192	186	31