CSMD: Assignment 5

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1 Three node bar element

1.1 Shape-functions coefficients

We can compute the coefficients of each shape-functions by substituting ξ in each N_i For $N_1(\xi) = a_0 + a_1\xi + a_2\xi^2$:

$$\begin{aligned} N_1(0) = 0 = a_0; \\ N_1(1) = 0 = a_1 + a_2; \\ N_1(-1) = 1 = a_2 - a_1; \\ a_0 = 0,\ a_1 = -\frac{1}{2},\ a_2 = \frac{1}{2} \end{aligned} \tag{1}$$

For $N_2(\xi) = b_0 + b_1\xi + b_2\xi^2$:

$$\begin{aligned} N_2(0) = 0 = b_0; \\ N_2(-1) = 0 = -b_1 + b_2; \\ N_2(1) = 1 = b_2 + b_1; \\ b_0 = 0,\ b_1 = \frac{1}{2},\ b_2 = \frac{1}{2} \end{aligned} \tag{2}$$

For $N_3(\xi) = c_0 + c_1\xi + c_2\xi^2$:

$$\begin{aligned} N_3(-1) = 0 = c_0 - c_1 + c_2; \\ N_3(1) = 0 = c_0 + c_1 + c_2; \\ N_3(0) = 1 = c_0; \\ c_0 = 1,\ c_1 = 0,\ c_2 = -1 \end{aligned} \tag{3}$$

The shape-functions then become $N_1(\xi) = -\frac{\xi}{2} + \frac{\xi^2}{2}, N_2(\xi) = \frac{\xi}{2} + \frac{\xi^2}{2}$ and $N_3(\xi) = 1 - \xi^2$.

1.2 Unity sum verification

It can be directly shown that the sum of the three shape-functions is 1:

$$N_1(\xi) + N_2(\xi) + N_3(\xi) = -\frac{\xi}{2} + \frac{\xi^2}{2} + \frac{\xi}{2} + \frac{\xi^2}{2} + 1 - \xi^2 = 1 \tag{4}$$

1.3 Shape-functions derivatives

Derivating each N_i with respect to the natural coordinate ξ:

$$\begin{aligned}\frac{\partial N_1}{\partial \xi} &= \xi - \frac{1}{2}\\ \frac{\partial N_2}{\partial \xi} &= \xi + \frac{1}{2}\\ \frac{\partial N_3}{\partial \xi} &= -2\xi\end{aligned} \tag{5}$$

2 The five node quadrilateral element

The fifth shape-function is the same as the nineth shape-function of the 9-node quadrilateral quadratic element, which is 1 in the central point and 0 in the sides. Its expression is

$$N_5 = (1-\xi^2)(1-\eta^2) \tag{6}$$

Following the hint, we make for i = 1:4, $N_i = \hat{N}_i + \alpha N_5$, where $\hat{N}_i$ is the 4-noded quadrilateral shape-function corresponding to node i. We can take any shape-function to compute α, as this parameter is the same for all of them. Substituting into N_1 yields:

$$\begin{aligned}N_1 = \frac{1}{4}(1-\xi)(1-\eta) + \alpha[(1-\xi^2)(1-\eta^2)];\\ N_1(0,0) = 0 = \frac{1}{4} + \alpha;\end{aligned} \tag{7}$$

Isolating α, we deduce its value: $\alpha = -\frac{1}{4}$. The sum of all shape-functions is, again, unity:

$$\begin{aligned}\sum_{i=1}^{3} N_i = \hat{N}_1 + \hat{N}_2 + \hat{N}_3 + \hat{N}_4 + N_5 + 4\alpha N_5 =\\ = 1 + (1-\xi^2)(1-\eta^2) - 4\frac{1}{4}(1-\xi^2)(1-\eta^2) = 1\end{aligned} \tag{8}$$

3 Integration of Gauss products

In order to ensure rank sufficiency for hexaedron numerical integration, we have to use enough Gauss points so as to have:

$$n_G \; n_E \geq n_F - n_R \tag{9}$$

with n_G, n_E, n_F, n_R being the number of Gauss points, order of the stress-strain matrix, number of independent rigid body motions and the element DoF, respectively. For a hexaedron of n nodes, the values of these variables are:

- $n_G =$ to be determined
- $n_E = 6$
- $n_R = 6$ (3 translations and 3 rotations)
- $n_F = 3n$

Next table shows the optimal (minimum) values of n_G:

n	8	20	27	64
n_F	24	60	81	192
n_R	6	6	6	6
n_G	8 (2x2x2)	27 (3x3x3)	27 (3x3x3)	64 (4x4x4)