COMPUTATIONAL SUCTURAL MECHANICS AND DYNAMICS

Master of Science in Computational Mechanics/Numerical Methods

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Assignment 5:

1. The isoparametric definition of the straight-node bar element in its local system $\bar{x}$ is:

$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \overline{x_1} & \overline{x_2} & \overline{x_3} \\ \overline{u_1} & \overline{u_2} & \overline{u_3} \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix}$$

Here ξ is the isoparametric coordinate that takes the values -1, 1 and 0 at nodes 1, 2 and 3 respectively, while N_1^e, N_2^e and N_3^e are the shape functions for a bar element. For simplicity, take $\overline{x_1} = 0$, $\overline{x_2} = L$, $\overline{x_3} = \frac{1}{2}L + \alpha L$. Here L is the bar length and α a parameter that characterizes how far node 3 is away from the midpoint location $\bar{x} = \frac{1}{2}L$. Show that the minimum α (minimal in absolute value sense) for which $J = d\bar{x}/d\xi$ vanishes at a point in the element are $\pm\frac{1}{4}$ (the quarter points). Interpret this result as a singularity by showing that the axial strain becomes infinite at an end point.

First of all, the trial functions are defined as the Lagrange interpolation functions of 2nd order:

$$N_1^e(\xi) = \frac{\xi(\xi-1)}{2}$$
$$N_2^e(\xi) = \frac{\xi(\xi+1)}{2}$$
$$N_3^e(\xi) = -(\xi+1)(\xi-1)$$

Using the isoparametric formulation, the geometry mapping is:

$$\bar{x} = \bar{x}_1 N_1^e(\xi) + \bar{x}_2 N_2^e(\xi) + \bar{x}_3 N_3^e(\xi)$$

And the Jacobian is defined as:

$$J = \frac{d\bar{x}}{d\xi} = \bar{x}_1 \frac{dN_1^e(\xi)}{d\xi} + \bar{x}_2 \frac{dN_2^e(\xi)}{d\xi} + \bar{x}_3 \frac{dN_3^e(\xi)}{d\xi}$$
$$J = \bar{x}_1\left(\xi - \frac{1}{2}\right) + \bar{x}_2\left(\xi + \frac{1}{2}\right) - 2\bar{x}_3\xi$$

Substituting the values for the nodal coordinates:

$$J = L\left(\xi + \frac{1}{2} - 2\xi\left(\frac{1}{2} + \alpha\right)\right) = L\left(\frac{1}{2} - 2\alpha\xi\right)$$

The critical value we are searching is when the Jacobian vanishes:

$$0 = L\left(\frac{1}{2} - 2\alpha^*\xi^*\right) \rightarrow \xi^* = \frac{1}{4\alpha^*}$$

The case we are interested is when this critical value fits inside the element domain:

$$-1 \le \xi^* \le 1 \rightarrow -1 \le \frac{1}{4\alpha^*} \le 1 \rightarrow -\frac{1}{4} \le \alpha^* \le \frac{1}{4}$$

In the case that $|\alpha| = \frac{1}{4}$, $\xi^* = \pm 1$. That means that the Jacobian vanishes at one or another end point. In this case, the axial strain become infinite at this point. This is easily shown as:

$$\frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx} = J^{-1}\frac{du}{d\xi}$$

2. **Extend the result obtained from the previous exercise for a 9-node plane stress element. The element is initially a perfect square, nodes 5, 6, 7, 8 are at the midpoint of the sides 1-2, 2-3, 3-4 and 4-1, respectively, and 9 at the centre of the square. Move node 5 tangentially towards 2 until the Jacobian determinant at 2 vanishes. This result is important in the construction of "singular elements" for fracture mechanics.**

The nodal coordinates are:

$$\boldsymbol{x_1} = \begin{bmatrix}-1\\-1\end{bmatrix}, \quad \boldsymbol{x_2} = \begin{bmatrix}1\\-1\end{bmatrix}, \quad \boldsymbol{x_3} = \begin{bmatrix}1\\1\end{bmatrix}, \quad \boldsymbol{x_4} = \begin{bmatrix}-1\\1\end{bmatrix}, \quad \boldsymbol{x_5} = \begin{bmatrix}\alpha\\-1\end{bmatrix},$$
$$\boldsymbol{x_6} = \begin{bmatrix}1\\0\end{bmatrix}, \quad \boldsymbol{x_7} = \begin{bmatrix}0\\1\end{bmatrix}, \quad \boldsymbol{x_8} = \begin{bmatrix}-1\\0\end{bmatrix}, \quad \boldsymbol{x_9} = \begin{bmatrix}0\\0\end{bmatrix}$$

The element shape functions are of the Lagrange family:

$$N_1^e = \frac{1}{4}(\xi - 1)(\eta - 1)\xi\eta$$
$$N_2^e = \frac{1}{4}(\xi + 1)(\eta - 1)\xi\eta$$
$$N_3^e = \frac{1}{4}(\xi + 1)(\eta + 1)\xi\eta$$
$$N_4^e = \frac{1}{4}(\xi - 1)(\eta + 1)\xi\eta$$
$$N_5^e = \frac{1}{2}(1 - \xi^2)\eta(\eta - 1)$$
$$N_6^e = \frac{1}{2}(1 - \eta^2)\xi(\xi + 1)$$
$$N_7^e = \frac{1}{2}(1 - \xi^2)\eta(\eta + 1)$$
$$N_8^e = \frac{1}{2}(1 - \eta^2)\xi(\xi - 1)$$
$$N_9^e = (1 - \xi^2)(1 - \eta^2)$$

The Jacobian is computed as:

$$J(\xi,\eta) = \sum_{i=1}^{9}\begin{bmatrix} x_i\frac{\partial N_i^e}{\partial \xi} & y_i\frac{\partial N_i^e}{\partial \xi} \\ x_i\frac{\partial N_i^e}{\partial \eta} & y_i\frac{\partial N_i^e}{\partial \eta}\end{bmatrix}$$

$$J_{11}(\xi,\eta) = -\frac{1}{4}(2\xi-1)(\eta-1)\eta + \frac{1}{4}(2\xi+1)(\eta-1)\eta + \frac{1}{4}(2\xi+1)(\eta+1)\eta - \frac{1}{4}(2\xi-1)(\eta+1)\eta - \alpha\xi\eta(\eta-1) + \frac{1}{2}(1-\eta^2)(2\xi+1) - \frac{1}{2}(1-\eta^2)(2\xi-1)$$

$$J_{12}(\xi,\eta) = -\frac{1}{4}(2\xi-1)(\eta-1)\eta - \frac{1}{4}(2\xi+1)(\eta-1)\eta + \frac{1}{4}(2\xi+1)(\eta+1)\eta + \frac{1}{4}(2\xi-1)(\eta+1)\eta + \xi\eta(\eta-1) - \xi\eta(\eta+1)$$

$$J_{21}(\xi,\eta) = -\frac{1}{4}(\xi-1)\xi(2\eta-1) + \frac{1}{4}(\xi+1)\xi(2\eta-1) + \frac{1}{4}(\xi+1)\xi(2\eta+1) - \frac{1}{4}(\xi-1)\xi(2\eta+1) + \frac{\alpha}{2}(1-\xi^2)(2\eta-1) - \eta\xi(\xi+1) + \eta\xi(\xi-1)$$

$$J_{22}(\xi,\eta) = -\frac{1}{4}(\xi-1)\xi(2\eta-1) - \frac{1}{4}(\xi+1)\xi(2\eta-1) + \frac{1}{4}(\xi+1)\xi(2\eta+1) + \frac{1}{4}(\xi-1)\xi(2\eta+1) - \frac{1}{2}(1-\xi^2)(2\eta-1) + \frac{1}{2}(1-\xi^2)(2\eta+1)$$

Evaluating the Jacobian matrix at the node 2:

$$J(1,-1) = \begin{bmatrix} 1-2\alpha & 0 \\ 0 & 1 \end{bmatrix}$$

The determinant is:

$$|J(1,-1)| = 1 - 2\alpha$$

The condition in order the Jacobian to vanish is $\alpha = \frac{1}{2}$. That is, the quarter point. This condition is the same than in the one dimensional case calculated in the previous case.