INTERNATIONAL CENTRE FOR
NUMERICAL METHODS IN ENGINEERING

UNIVERSITAT POLITÈCNICA DE CATALUNYA

MASTER OF SCIENCE IN COMPUTATIONAL MECHANICS

Computational Structural Mechanics and Dynamics

Assignment 5

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ASSIGNMENT 5.1

1. *The isoparametric definition of the straight–node bar element in its local system $\underline{x}$ is,*

$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix} \tag{0.1}$$

Here ξ is the isoparametric coordinate that takes the values -1, 1 and 0 at nodes 1, 2 and 3 respectively, while N_1^e, N_2^e and N_3^e are the shape functions for a bar element.

For simplicity, take $\bar{x}_1 = 0$, $\bar{x}_2 = l$, $\bar{x}_3 = \frac{1}{2}l + \alpha l$. Here l is the bar length and α a parameter that characterizes how far node 3 is away from the midpoint location $\bar{x} = \frac{1}{2}l$.

Show that the minimum a (minimal in absolute value sense) for which $J = d\bar{x}/d\xi$ vanishes at a point in the element are $\pm 1/4$ (the quarter points). Interpret this result as a singularity by showing that the axial strain becomes infinite at an end point.

Solution 1:

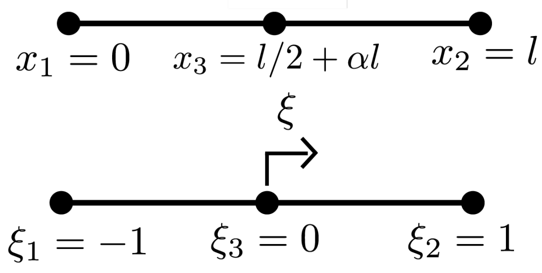

Figure 0.1: 1D Isoparametric element.

As mentioned in the definition of the problem, the axial displacement is expressed by

$$u = N_1(\xi)u_1 + N_2(\xi)u_2 + N_3(\xi)u_3 \tag{0.2}$$

and the x coordinate of a point within the element is written as well in the parametric formulation as:

$$x = N_1(\xi)x_1 + N_2(\xi)x_2 + N_3(\xi)x_3 \tag{0.3}$$

where the shape functions $N_1(\xi)$, $N_2(\xi)$ and $N_3(\xi)$ are obtained specifically for the given problem as the equation 0.4, and also can be verified in the figure 0.1.
For the given element, the shape functions are:

$$\begin{aligned} N_1 &= \frac{1}{2}\xi(\xi-1) \\ N_2 &= \frac{1}{2}\xi(\xi+1) \\ N_3 &= 1-\xi^2 \end{aligned} \tag{0.4}$$

Now, the axial strain is obtained by:

$$\epsilon = \frac{du}{dx} = \sum_{i=1}^{3} \frac{dN_i}{d\xi} u_i = \left[\frac{\partial N_1}{\partial \xi}\frac{\partial \xi}{\partial x}, \frac{\partial N_2}{\partial \xi}\frac{\partial \xi}{\partial x}, \frac{\partial N_3}{\partial \xi}\frac{d\xi}{\partial x} \right] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{0.5}$$

Applying the derivatives of each shape equation respect to ξ:

$$\frac{\partial N_1}{\partial \xi} = \xi - \frac{1}{2}, \qquad \frac{\partial N_2}{\partial \xi} = \xi + \frac{1}{2}, \qquad \frac{\partial N_3}{\partial \xi} = -2\xi$$

Now, considering the derivative of the isoparametric formulation of the x coordinate:

$$\begin{aligned} \frac{\partial x}{\partial \xi} &= \frac{\partial N_1}{\partial \xi}x_1 + \frac{\partial N_2}{\partial \xi}x_2 + \frac{\partial N_3}{\partial \xi}x_3 \\ &= \left(\xi - \frac{1}{2}\right)(0) + \left(\xi + \frac{1}{2}\right)(l) + (-2\xi)\left(\frac{l}{2} + \alpha l\right) \\ &= \frac{l}{2}(\xi+1) + \frac{l}{2}\xi + \left(\frac{l}{2} + \alpha l\right)(-2\xi) \\ &= \frac{l}{2} - 2\alpha l \xi = \frac{1}{2}(l - 4\alpha l \xi) \end{aligned}$$

The equation (which corresponds to the Jacobian of 1D system) that provides a relationship between dx and $d\xi$ in terms of the three nodal coordinates is:

$$\frac{\partial \xi}{\partial x} = \frac{2}{l - 4\alpha l \xi} \tag{0.6}$$

As seen in the above equation, there is a singularity when the Jacobian is computed at two points, which corresponds to each end of the bar. In that sense, the strains computed at the first and second node are:

$$\frac{\partial N_1}{\partial \xi}\frac{\partial \xi}{\partial x} = \left(\xi - \frac{1}{2}\right)\left(\frac{2}{l - 4\alpha l \xi}\right) = \left((-1) - \frac{1}{2}\right)\left(\frac{2}{l - 4\alpha l(-1)}\right) = \frac{-3}{l(1+4\alpha)}$$

$$\frac{\partial N_2}{\partial \xi}\frac{\partial \xi}{\partial x} = \left(\xi + \frac{1}{2}\right)\left(\frac{2}{l - 4\alpha l \xi}\right) = \left((1) + \frac{1}{2}\right)\left(\frac{2}{l - 4\alpha l(1)}\right) = \frac{3}{l(1-4\alpha)}$$

In which it can be observed that the two critical values are:

- Node 1: $\xi = -1$ with $\alpha = -1/4$
- Node 2: $\xi = 1$ with $\alpha = 1/4$

In conclusion, for $\alpha = \pm 1/4$ at the end points, the strain value becomes infinite.

ASSIGNMENT 5.2

1. *Extend the results obtained from the previous Exercise for a 9-node plane stress element. The element is initially a perfect square, nodes 5, 6 ,7 8 are at the midpoint of the sides 1-2, 2-3, 3-4 and 4-1, respectively, and 9 at the center of the square.*

Move node 5 tangentially towards 2 until the Jacobian determinant at 2 vanishes. This result is important in the construction of singular elements for fracture mechanics.

Solution 2:

Considering a plane stress quadrilateral element with 9 nodes, which located as the picture 0.2 shows:

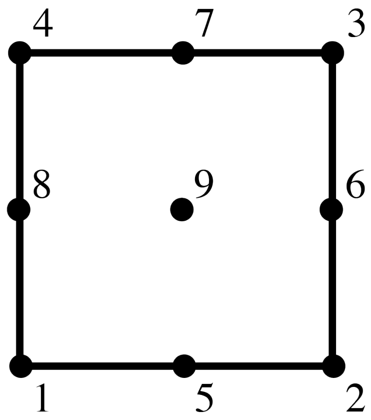

Figure 0.2: 2D Isoparametric quadrilateral element.

Following a similar procedure than the 1D case, the first step is the formulation of the approximate solution and the geometry, which can be obtained in terms of the known shape functions as:

$$x = \sum_{i=1}^{n} x_i N_i \qquad y = \sum_{i=1}^{n} y_i N_i$$

As working in 2D, there is more complexity of the formulation of the shape functions, in that sense the procedure followed is named as line-product method, for example the shape function of the ninth node is obtained as:

$$N_9 = c_9 L_{1-2} L_{2-3} L_{3-4} L_{4-1} \tag{0.7}$$

Then, the shape functions of each node are:

- **Node 1:** $N_1 = \frac{1}{4}\xi\eta(\xi-1)(\eta-1)$
- **Node 2:** $N_2 = \frac{1}{4}\xi\eta(\xi+1)(\eta-1)$
- **Node 3:** $N_3 = \frac{1}{4}\xi\eta(\xi+1)(\eta+1)$
- **Node 4:** $N_4 = \frac{1}{4}\xi\eta(\xi-1)(\eta+1)$
- **Node 5:** $N_5 = \frac{1}{2}\eta(1-\xi^2)(\eta-1)$
- **Node 6:** $N_6 = \frac{1}{2}\xi(\xi+1)(1-\eta^2)$
- **Node 7:** $N_7 = \frac{1}{2}\eta(1-\xi^2)(\eta+1)$
- **Node 8:** $N_8 = \frac{1}{2}\xi(\xi-1)(1-\eta^2)$
- **Node 9:** $N_9 = (1-\xi^2)(1-\eta^2)$

Differentiating with respect to the quadrilateral coordinates:

$$\frac{\partial x}{\partial \xi} = \sum_{i=1}^{n} x_i \frac{\partial N_i}{\partial \xi}, \qquad \frac{\partial y}{\partial \xi} = \sum_{i=1}^{n} y_i \frac{\partial N_i}{\partial \xi}, \qquad \frac{\partial x}{\partial \eta} = \sum_{i=1}^{n} x_i \frac{\partial N_i}{\partial \eta}, \qquad \frac{\partial y}{\partial \eta} = \sum_{i=1}^{n} y_i \frac{\partial N_i}{\partial \eta},$$

These expressions can be seen in matrix form as:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix} \tag{0.8}$$

The entries of the above matrix conform the Jacobian of the system that provides the relationship between the system of coordinates which is obtained inverting this matrix. In order to complete the components of this matrix, it is needed to compute the derivatives of the shape functions:

- **Node 1:** $\frac{\partial N_1}{\partial \xi} = \frac{1}{4}\eta(2\xi-1)(\eta-1)$ $\quad \frac{\partial N_1}{\partial \eta} = \frac{1}{4}\xi(\xi-1)(2\eta-1)$
- **Node 2:** $\frac{\partial N_2}{\partial \xi} = \frac{1}{4}\eta(2\xi+1)(\eta-1)$ $\quad \frac{\partial N_2}{\partial \eta} = \frac{1}{4}\xi(\xi+1)(2\eta-1)$
- **Node 3:** $\frac{\partial N_3}{\partial \xi} = \frac{1}{4}\eta(2\xi+1)(\eta+1)$ $\quad \frac{\partial N_3}{\partial \eta} = \frac{1}{4}\xi(\xi+1)(2\eta+1)$
- **Node 4:** $\frac{\partial N_4}{\partial \xi} = \frac{1}{4}\eta(2\xi-1)(\eta+1)$ $\quad \frac{\partial N_4}{\partial \eta} = \frac{1}{4}\xi(\xi-1)(2\eta+1)$
- **Node 5:** $\frac{\partial N_5}{\partial \xi} = -\xi\eta(\eta-1)$ $\quad \frac{\partial N_5}{\partial \eta} = \frac{1}{2}(1-\xi^2)(2\eta-1)$
- **Node 6:** $\frac{\partial N_6}{\partial \xi} = \frac{1}{2}(2\xi+1)(1-\eta^2)$ $\quad \frac{\partial N_6}{\partial \eta} = -\xi\eta(\xi+1)$
- **Node 7:** $\frac{\partial N_7}{\partial \xi} = -\xi\eta(\eta+1)$ $\quad \frac{\partial N_7}{\partial \eta} = \frac{1}{2}(1-\xi^2)(2\eta+1)$
- **Node 8:** $\frac{\partial N_8}{\partial \xi} = \frac{1}{2}(2\xi-1)(1-\eta^2)$ $\quad \frac{\partial N_8}{\partial \eta} = -\xi\eta(\xi-1)$
- **Node 9:** $\frac{\partial N_9}{\partial \xi} = -2\xi(1-\eta^2)$ $\quad \frac{\partial N_9}{\partial \eta} = -2\eta(1-\xi^2)$

Now, computing the jacobian and substituting the value of the node 2 which corresponds in the normalized system of coordinates as $(\xi, \eta) = (1, -1)$:

$$\mathbf{J}(\xi,\eta) = \begin{bmatrix} \frac{l}{2} - 2a & 0 \\ 0 & \frac{l}{2} \end{bmatrix}$$

Then, the relationship of the system of coordinates requires the determinant of the Jacobian, in that sense the computing reduces to the next value:

$$|J| = 0$$
$$\frac{l^2}{4} - l\alpha = 0$$
$$\alpha = \frac{l}{4}$$

Which corresponds to a result that is compared with the 1D element, so the conclusion is evaluated and demonstrated to be equivalently in the 2D quadrilateral 9-node element.