Universitat Politècnica de Catalunya

Computational Solid Mechanics and Dynamics

Master's Degree in Numerical Methods in Engineering

On 'Convergence Requirements'

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1 Assignment 5.1

If writing a relation between the coordinates of the nodes of an element and the Cartesian coordinates, it is obtained that

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 \tag{1}$$

Therefore the shape functions are chosen such that N_i must be a function which is unity at the nodes. Specifically, the shape functions corresponding to three nodes are

$$\begin{cases} N_1 = \frac{\xi(\xi-1)}{2} \\ N_2 = \frac{\xi(\xi+1)}{2} \\ N_3 = -(\xi-1)(\xi+1) \end{cases} \tag{2}$$

Then, using the nodal coordinates in the local system (where $x_1 = 0$)

$$x = \frac{\xi(\xi+1)}{2} l - (\xi-1)(\xi+1)(\frac{l}{2} + \alpha l) = \frac{l}{2}(1+\xi)(2\alpha - 2\xi\alpha + 1) \tag{3}$$

It is possible to see from equation (3) that the coordinates of the nodes are obtained if the value of ξ are substituted. Now the derivative of (3) with respect to ξ is easily calculated.

$$\frac{dx}{d\xi} = \frac{l}{2}(1 - 4\alpha\xi) \tag{4}$$

Equation (4) is clearly the Jacobian. The value of the Jacobian is positive on the whole domain when $1/4 < \alpha < 1/4$, and to prove that equation (4) may be plotted for the extreme cases $\xi = 1, -1$ as a function of alpha. Figure 1 shows that the jacobian is always positive for this range of α, being it null at the extreme values. This proves that the minimum α in absolute value that makes the Jacobian zero is $\alpha = 1/4$ when $\xi = 1$ and $\alpha = -1/4$ when $\xi = -1$, that is, the extreme values. This may be directly seen from (4).

The strain displacement matrix is simply calculated as

$$\mathbf{B} = J^{-1}\frac{d\mathbf{N}}{d\xi} = \frac{2}{1-4\alpha\xi}\left[\xi - 1/2, \quad \xi + 1/2, \quad -2\xi\right] \tag{5}$$

Now, equation (5) allows us to define the quarter points as a singularity by noticing the following

- For $\xi = 1 \longrightarrow \lim_{\alpha \to 1/4} \mathbf{B} = [\infty \quad \infty \quad \infty]$
- For $\xi = -1 \longrightarrow \lim_{\alpha \to -1/4} \mathbf{B} = [\infty \quad \infty \quad \infty]$

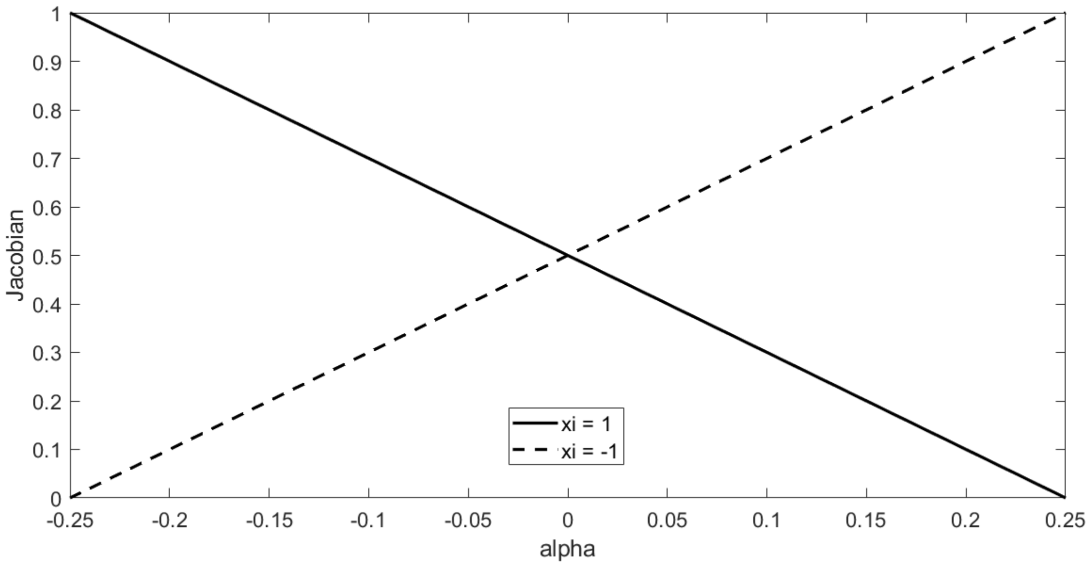

Figure 1: Variation of the Jacobian at the extreme points for various values of α.

2 Assignment 5.2

Now we are focused in analyzing a nine-noded quadratic Lagrange rectangle with the nodes numbered as in Fig. 2.

Of course as a condition for completeness the shape functions must sum one at each node, and zero at the others. The shape functions are described as

$$\begin{cases} N_l = \frac{1}{4}(\xi^2 + \overline{\xi}_l)(\eta^2 + \overline{\eta}_l), \quad l = 1,3,5,7 \\ N_l = \frac{1}{2}\eta_l^2(\eta^2 - \overline{\eta}_l)(1-\xi^2) + \frac{1}{2}\xi_l^2(\xi^2 - \overline{\xi}_l)(1-\eta^2), \quad l = 2,4,6,8 \\ N_l = (1-\eta^2)(1-\xi^2), \quad l = 9 \end{cases} \tag{6}$$

With $\overline{\xi}_l = \xi\xi_l$, $\overline{\eta}_l = \eta\eta_l$. The nine nodes have the local coordinates shown in Table 1.

Again, if we consider l to be the length of the square, the position of node 5 may be defined as $\mathbf{x_5} = [l/2 + \alpha l, 0]$ if we consider the node 1 to be the origin of the global coordinates. Now the procedure to follow which will be the value of α such that the determinant when $\xi = 1, \eta = -1$ vanishes. For that we need to compute the Jacobian first.

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi}; \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{7}$$

The purpose is to compute the Jacobian at node 2. For this a Matlab code is in order which will simplify the task of calculating the derivatives. The coordinates of the nodes are

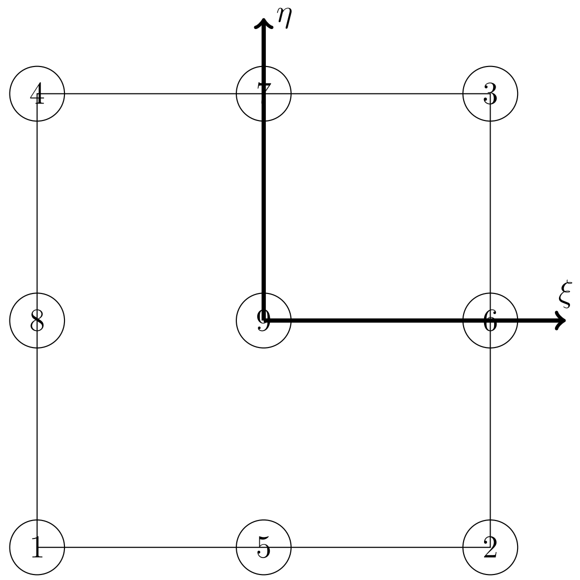

Figure 2: Local numbering of the element.

$$\mathbf{X} = \begin{bmatrix} \mathbf{x_i} \\ \mathbf{y_i} \end{bmatrix} \quad (i = 1, 2, ..., 9) \quad = \begin{bmatrix} 0 & l & l & 0 & l/2 + \alpha l & l & l/2 & 0 & l \\ 0 & 0 & l & l & 0 & l/2 & l & l/2 & l/2 \end{bmatrix}^T \tag{8}$$

Now,

$$x = \sum_{i=0}^{9} x_i N_i, \quad y = \sum_{i=0}^{9} y_i N_i \tag{9}$$

Eventually, the process is:

- Compute the shape functions with Table 1 and equation (6).
- Compute the mapping with equation (9) and the nodal coordinates in equation (8).
- Compute the jacobian with equation (7) and substitute the particular local values of node 2, i.e. $\xi = 1, \eta = -1$.
- Find the value of α that minimizes the value of the determinant of the jacobian.

Eventually, once the operations have been computed (see Matlab code attached) the determinant of the Jacobian at the second node is equal to

$$det(\mathbf{J}) = \frac{l^2}{4} - \alpha l^2 \longrightarrow det(\mathbf{J}) = 0 \longrightarrow \alpha = 1/4 \tag{10}$$

Node	ξ_i	η_i
1	-1	-1
2	0	-1
3	1	-1
4	1	0
5	1	1
6	0	1
7	-1	1
8	-1	0
9	0	0

Table 1: Nodal local coordinates.

With this value of α, the coordinate of the node 5 would be $\mathbf{x_5} = [l/2 + \alpha l, 0] = [l/2 + \frac{l}{4}, 0] = [\frac{3l}{4}, 0]$, which is actually the third quarter of the side 1-2. This is comparable with what was found with the triangular case in which the value of alpha was 1/4.

A Appendix:Code

```
syms xi eta alpha l

%% SHAPE FUNCTIONS
N{1} = (xi-1)*(eta-1)*xi*eta/4;%ok
N{2} = (xi+1)*(eta-1)*xi*eta/4;%ok
N{3} = (xi+1)*(eta+1)*xi*eta/4;%ok
N{4} = (xi-1)*(eta+1)*xi*eta/4;%ok
N{5} = (xi+1)*(xi-1)*eta*(1 - eta)/2; %ok
N{6} = (xi+1)*xi*(eta+1)*(1-eta)/2; %ok
N{7} = -(xi+1)*(xi-1)*eta*(1+eta)/2;%ok
N{8} = -(xi-1)*xi*(eta-1)*(eta+1)/2;
N{9} = (1-xi^2)*(1-eta^2); %ok

%% COORDINATES OF THE NODES
C = [0 0; ...
    l 0;...
    l l;...
    0 l; ...
    l/2 + alpha*l 0;...
    l l/2;...
    l/2 l;...
    0 l/2; ...
    l/2 l/2];

%% PARAMETRIC MAPPING
sx = N{1}*C(1,1) + N{2}*C(2,1) + N{3}*C(3,1) + N{4}*C(4,1) + ...
    N{5}*C(5,1) + N{6}*C(6,1) + N{7}*C(7,1) + N{8}*C(8,1) + N{9}*C(9,1);
sy = N{1}*C(1,2) + N{2}*C(2,2) + N{3}*C(3,2) + N{4}*C(4,2) + ...
    N{5}*C(5,2) + N{6}*C(6,2) + N{7}*C(7,2) + N{8}*C(8,2) + N{9}*C(9,2);

%% COMPUTATION OF THE JACOBIAN
A = diff(sx,xi); B = diff(sy,xi);
C = diff(sx,eta); D = diff(sy,eta);

J = [A B; C D]; detJ = det(J);

%% COMPUTATION OF THE DETERMINANT OF THE JACOBIAN
detJ = subs(detJ,xi,1);
detJ = subs(detJ,eta,-1);
a = solve(detJ == 0,alpha); %Value that we seek-> alpha = 1/4
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References

[1] O.C. Zienkiewics and K. Morgan, *Finite Elements and Approximation*. Dover books ISNB 0-486-4530149 (1983).