Universitat Politecnica de Catalunya CIMNE
Master in Computational Mechanics

Computational Structural Mechanics & Dynamics

Assignment 5

Convergence Requirements

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16th March 2020

Assignment 5.1 :

The isoparametric definition of the straight–node bar element in its local system x is,

$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix}$$

Here ξ is the isoparametric coordinate that takes the values –1, 1 and 0 at nodes 1, 2 and 3 respectively, while N^e_1, N^e_2 and N^e_3 are the shape functions for a bar element.

For simplicity, take $\bar{x}_1 = 0, \bar{x}_2 = L, \bar{x}_3 = \frac{1}{2}l + \alpha l$. Here l is the bar length and α a parameter that characterizes how far node 3 is away from the midpoint location $\bar{x} = \frac{1}{2}l$.

Show that the minimum α (minimal in absolute value sense) for which $J = d\bar{x}/d\xi$ vanishes at a point in the element are ±¼ (the quarter points). Interpret this result as a singularity by showing that the axial strain becomes infinite at an end point.

Solution:

$x_1 = 0\,, x_2 = l\,, x_3 = \left(\frac{1}{2} + \alpha\right) l$ and $\xi_1 = 1\,, \xi_2 = -1\,, \xi_3 = 0$

The Shape functions for 1D bar element with 3 nodes are:

$$N_1 = \frac{(\xi_2-\xi)(\xi_3-\xi)}{(\xi_2-\xi_1)(\xi_3-\xi_1)}\,, N_2 = \frac{(\xi_3-\xi)(\xi_1-\xi)}{(\xi_3-\xi_2)(\xi_1-\xi_2)}\,, N_3 = \frac{(\xi_1-\xi)(\xi_3-\xi)}{(\xi_1-\xi_3)(\xi_2-\xi_3)}$$

So, after substituting the values, the shape functions become:

$$N_1 = \frac{\xi(\xi-1)}{2}\,, \quad N_2 = \frac{\xi(\xi+1)}{2}, \quad N_3 = 1 - \xi^2$$

$$x = N_1(\xi)x_1 + N_2(\xi)x_2 + N_3(\xi)x_3$$

So, $$J = \frac{dx}{d\xi} = \frac{dN_1}{d\xi}x_1 + \frac{dN_2}{d\xi}x_2 + \frac{dN_3}{d\xi}x_3$$

$$= 0 + \left(\frac{1}{2} + \xi\right) l - 2\xi(\frac{1}{2} + \alpha)l$$

$$= 0 + \left(\frac{1}{2} + \xi\right) l - 2\xi(\frac{1}{2} + \alpha)l$$

$$= \left(\frac{l}{2}\right) + \xi l - \xi l - 2\xi\alpha l$$

$$J = \left(\frac{1}{2} - 2\xi\alpha\right) l$$

- For J to vanish, the value of J should be zero. Hence, equating the the above obtained equation of J to zero, we get:

$$J = \left(\frac{1}{2} - 2\xi\alpha\right) l = 0$$

$$\therefore \left(\frac{1}{2} - 2\xi\alpha\right) l = 0$$

$$\therefore \ \frac{1}{2} = 2\xi\alpha$$

From this, we get that $\boxed{\alpha = \ \pm\frac{1}{4}}$

Hence, the minimum value of α for which the Jacobian vanishes is $\alpha = \ \pm\frac{1}{4}$

The axial strain equation is given by,

$$B = \frac{dN}{dx} = J^{-1}\frac{dN}{d\xi}$$

$$B = \frac{1}{\left(\frac{1}{2} - 2\xi\alpha\right)l}\begin{bmatrix}\frac{dN_1}{d\xi} & \frac{dN_2}{d\xi} & \frac{dN_3}{d\xi}\end{bmatrix}$$

$$B = \frac{1}{\left(\frac{1}{2} - 2\xi\alpha\right)l}\left[(\xi - \frac{1}{2}), \quad \left(\xi + \frac{1}{2}\right), \quad -2\xi\right]$$

But, we know that at $\alpha = \pm\frac{1}{4}$ the Jacobian becomes zero, hence the axial strain tends to infinity as shown below,

$$B = \frac{1}{0}\left[(\xi - \frac{1}{2}), \quad \left(\xi + \frac{1}{2}\right), \quad -2\xi\right]$$

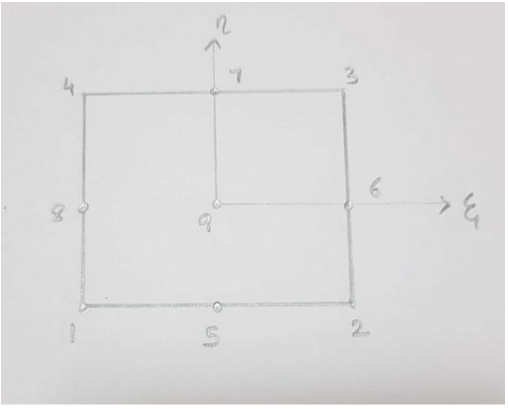
$$B = \infty\left[(\xi - \frac{1}{2}), \quad \left(\xi + \frac{1}{2}\right), \quad -2\xi\right]$$

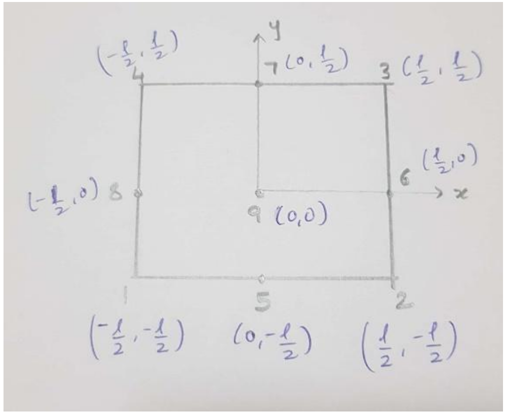
Thus, the strain becomes infinite at the end points when we substitute the values $\xi = \ \pm 1$ and $\alpha = \pm\frac{1}{4}$

Assignment 5.2 :

Extend the results obtained from the previous Exercise for a 9-node plane stress element. The element is initially a perfect square, nodes 5,6,7,8 are at the midpoint of the sides 1–2, 2–3, 3–4 and 4–1, respectively, and 9 at the center of the square.

Move node 5 tangentially towards 2 until the Jacobian determinant at 2 vanishes. This result is important in the construction of "singular elements" for fracture mechanics.





Solution:

The isoparametric definition of this element is given by,

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} & u_{x7} & u_{x8} & u_{x9} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} & u_{y7} & u_{y8} & u_{y9} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \\ N_7 \\ N_8 \\ N_9 \end{bmatrix}$$

The Shape functions for 9 noded quadrilateral element are given by,

$N_1 = \frac{1}{4}(\xi - 1)(\eta - 1)\xi\eta$ $\quad N_5 = \frac{1}{2}(1 - \xi^2)(\eta - 1)\eta$ $\quad N_9 = (1 - \xi^2)(1 - \eta^2)$

$N_2 = \frac{1}{4}(\xi + 1)(\eta - 1)\xi\eta$ $\quad N_6 = \frac{1}{2}(\xi + 1)(1 - \eta^2)\xi$

$N_3 = \frac{1}{4}(\xi + 1)(\eta + 1)\xi\eta$ $\quad N_7 = \frac{1}{2}(1 - \xi^2)(\eta + 1)\eta$

$N_4 = \frac{1}{4}(\xi - 1)(\eta + 1)\xi\eta$ $\quad N_8 = \frac{1}{2}(\xi - 1)(1 - \eta^2)\xi$

The derivatives of these shape functions respectively are,

$\frac{\partial N_1}{\partial \xi} = \frac{1}{4}(2\xi - 1)(\eta - 1)\eta$ $\quad \frac{\partial N_1}{\partial \eta} = \frac{1}{4}(2\eta - 1)(\xi - 1)\xi$

$\frac{\partial N_2}{\partial \xi} = \frac{1}{4}(2\xi + 1)(\eta - 1)\eta$ $\quad \frac{\partial N_2}{\partial \eta} = \frac{1}{4}(\xi + 1)(2\eta - 1)\xi$

$\frac{\partial N_3}{\partial \xi} = \frac{1}{4}(2\xi + 1)(\eta + 1)\eta$ $\quad \frac{\partial N_3}{\partial \eta} = \frac{1}{4}(\xi + 1)(2\eta + 1)\xi$

$\frac{\partial N_4}{\partial \xi} = \frac{1}{4}(2\xi - 1)(\eta + 1)\eta$ $\quad \frac{\partial N_4}{\partial \eta} = \frac{1}{4}(\xi - 1)(2\eta + 1)\xi$

$\frac{\partial N_5}{\partial \xi} = -\xi\eta(\eta - 1)$ $\quad \frac{\partial N_5}{\partial \eta} = \frac{1}{2}(2\eta - 1)\left(1 - \xi^2\right)$

$\frac{\partial N_6}{\partial \xi} = \frac{1}{2}(2\xi+1)(1-\eta^2)$ $\frac{\partial N_6}{\partial \eta} = -\xi\eta(\xi+1)$

$\frac{\partial N_7}{\partial \xi} = -\xi\eta(\eta+1)$ $\frac{\partial N_7}{\partial \eta} = \frac{1}{2}(2\eta+1)\left(1-\xi^2\right)$

$\frac{\partial N_8}{\partial \xi} = \frac{1}{2}(2\xi-1)(1-\eta^2)$ $\frac{\partial N_8}{\partial \eta} = -\xi\eta(\xi-1)$

$\frac{\partial N_9}{\partial \xi} = -2\xi(1-\eta^2)$ $\frac{\partial N_9}{\partial \eta} = -2\eta\left(1-\xi^2\right)$

The Jacobian is given by,

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$$\frac{\partial x}{\partial \xi} = \sum_{i=1}^{9} x_i \frac{\partial N_i}{\partial \xi}, \quad \frac{\partial y}{\partial \xi} = \sum_{i=1}^{9} y_i \frac{\partial N_i}{\partial \xi}, \quad \frac{\partial x}{\partial \eta} = \sum_{i=1}^{9} x_i \frac{\partial N_i}{\partial \eta}, \quad \frac{\partial y}{\partial \eta} = \sum_{i=1}^{9} y_i \frac{\partial N_i}{\partial \eta}$$

The co-ordinates of x and y at different nodes for the given element are,

Nodes	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
x	$-\frac{l}{2}$	$\frac{l}{2}$	$\frac{l}{2}$	$-\frac{l}{2}$	0	$\frac{l}{2}$	0	$-\frac{l}{2}$	0
y	$-\frac{l}{2}$	$-\frac{l}{2}$	$\frac{l}{2}$	$\frac{l}{2}$	$-\frac{l}{2}$	0	$\frac{l}{2}$	0	0

So, solving for node 2, after substituting the respective values of derivatives of shape functions, and the x and y co-ordinate values, with $\xi = 1\ and\ \eta = -1$, the Jacobian becomes,

$$J = \begin{bmatrix} \sum_{i=1}^{9} x_i \frac{\partial N_i}{\partial \xi} & \sum_{i=1}^{9} y_i \frac{\partial N_i}{\partial \xi} \\ \sum_{i=1}^{9} x_i \frac{\partial N_i}{\partial \eta} & \sum_{i=1}^{9} y_i \frac{\partial N_i}{\partial \eta} \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{l}{2} - 2\alpha l & 0 \\ 0 & \frac{l}{2} \end{bmatrix}$$

For the Jacobian to vanish, the determinant should become zero, so equating the determinant to zero,

$$\left(\frac{l}{2} - 2\alpha l\right) * \left(\frac{l}{2}\right) = 0$$

$$\therefore \left(\frac{l^2}{4} - \alpha l^2\right) = 0$$

$$\therefore \alpha = \frac{1}{4}$$

So, we can observe that the value of α comes to be $\frac{1}{4}$ when the Jacobian reduces to 0. This is same as the one done in earlier exercise.

Thus, in a quadratic element, when the middle node is at a distance of $\frac{1}{4}$ from its end nodes, the Jacobian becomes singular and thus vanishes.

Discussions:

- The main idea of the Jacobian is that, it relates the natural co-ordinates of a geometry with the computational co-ordinates. The Jacobian should be positive always in order for the mapping to exist. The Jacobian becomes singular at quarter points between 2 nodes for a 3 noded quadratic bar element.
- The important parameters required for convergence are consistency (completeness) and stability (positive Jacobian). Both of which are reflected in the above exercise. The sum of all shape functions equalling 1 establishes completeness and the positive value of Jacobian ensures that the solution is valid and stable.