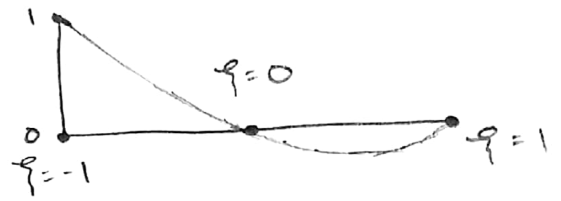
Problem 5.1a

we will first solve for the coefficients $a_0, \dots c_2$ using the node value conditions given

$$N_1^e(\xi) = a_0 + a_1\xi + a_2\xi^2$$

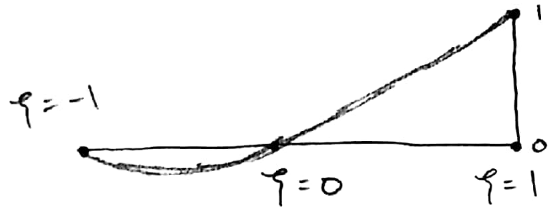


$$N_1^e(-1) = a_0 + a_1(-1) + a_2(-1)^2 = 1$$
$$N_1^e(0) = a_0 = 0$$
$$N_1^e(1) = a_0 + a_1(1) + a_2(1)^2 = 0$$

$$a_1 + a_2 = 0$$
$$a_2 - a_1 = 1$$
$$2a_2 = 1$$

$$\boxed{a_0 = 0 \quad a_1 = -1/2 \quad a_2 = 1/2}$$

$$N_2^e(\xi) = b_0 + b_1\xi + b_2\xi^2$$

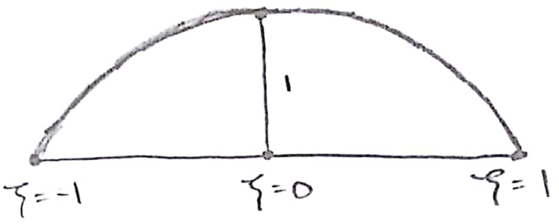


$$N_2^e(-1) = b_0 + b_1(-1) + b_2(-1)^2 = 0$$
$$N_2^e(0) = b_0 = 0$$
$$N_2^e(1) = b_0 + b_1(1) + b_2(1)^2 = 1$$

$$b_2 - b_1 = 0$$
$$b_1 + b_2 = 1$$

$$\boxed{b_0 = 0 \quad b_1 = 1/2 \quad b_2 = 1/2}$$

$$N_3^e(\xi) = c_0 + c_1(\xi) + c_2(\xi)^2$$



$$N_3^e(-1) = c_0 + c_1(-1) + c_2(-1)^2 = 0$$

$$N_3^e(0) = c_0 = 1$$

$$N_3^e(1) = c_0 + c_1(1) + c_2(1)^2 = 0$$

$$c_1 + c_2 = -1$$

$$-c_1 + c_2 = -1$$

$$\boxed{\begin{aligned} c_0 &= 1 \\ c_1 &= 0 \\ c_2 &= -1 \end{aligned}}$$

Problem 5.1b Now, verifying that $N_1 + N_2 + N_3 = 1$

$$N_1^e = -\frac{1}{2}\xi + \frac{1}{2}\xi^2$$

$$N_2^e = \frac{1}{2}\xi + \frac{1}{2}\xi^2$$

$$N_3^e = 1 - \xi^2$$

$$N_1 + N_2 + N_3 = \cancel{-\frac{1}{2}\xi} + \cancel{\frac{1}{2}\xi^2} + \cancel{\frac{1}{2}\xi} + \cancel{\frac{1}{2}\xi^2} + 1 \cancel{- \xi^2}$$

$$\boxed{N_1 + N_2 + N_3 = 1}$$

Problem 5.1c

$$N_1^e = -\frac{1}{2}\xi + \frac{1}{2}\xi^2$$

$$N_2^e = \frac{1}{2}\xi + \frac{1}{2}\xi^2$$

$$N_3^e = 1 - \xi^2$$

Now taking the derivitives with respect to (ξ)

$$\boxed{\begin{aligned} \frac{dN_1^e}{d\xi} &= -\frac{1}{2} + \xi \\ \frac{dN_2^e}{d\xi} &= \frac{1}{2} + \xi \\ \frac{dN_3^e}{d\xi} &= -2\xi \end{aligned}}$$

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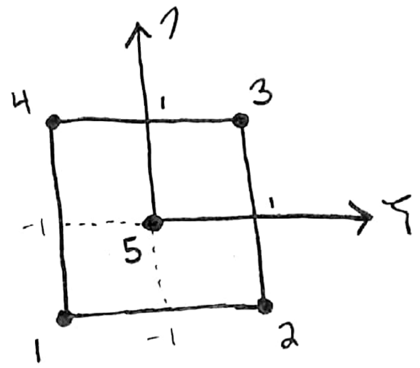
Problem 5.2

For the 5-noded Quadrilateral element



we can develop N_5 such that

$$N_5 = (1-\xi^2)(1-\eta^2)$$

and developing $\underline{N_i}$ $i=1,2,3,4$ gives us

$$\underline{N_1} = \frac{1}{4}(1-\xi)(1-\eta)$$

$$\underline{N_2} = \frac{1}{4}(1+\xi)(1-\eta)$$

$$\underline{N_3} = \frac{1}{4}(1+\xi)(1+\eta)$$

$$\underline{N_4} = \frac{1}{4}(1-\xi)(1+\eta)$$

Now, combining and solving for α as such gives us

$$N_i = \underline{N_i} + \alpha N_5$$

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) + \alpha(1-\xi^2)(1-\eta^2)$$

at $\xi = 0$ & $\eta = 0$

$$N_1 = \frac{1}{4} + \alpha \quad \therefore \quad \boxed{\alpha = -1/4}$$

Now we will generate all 5 shape functions using $\alpha = -1/4$

$$N_1 = \frac{1}{4}\left[(1-\xi)(1-\eta) - (1-\xi^2)(1-\eta^2)\right]$$

$$N_2 = \frac{1}{4}\left[(1+\xi)(1-\eta) - (1-\xi^2)(1-\eta^2)\right]$$

$$N_3 = \frac{1}{4}\left[(1+\xi)(1+\eta) - (1-\xi^2)(1-\eta^2)\right]$$

$$N_4 = \frac{1}{4}\left[(1-\xi)(1+\eta) - (1-\xi^2)(1-\eta^2)\right]$$

$$N_5 = (1-\xi^2)(1-\eta^2)$$

we will now check that $N_1 + N_2 + N_3 + N_4 + N_5 = 1$

$$N_1 + N_2 + N_3 + N_4 + N_5 = \frac{1}{4}(1-\xi-\eta+\xi\eta) + \frac{1}{4}(1+\xi-\eta-\xi\eta) \ldots$$

$$\ldots + \frac{1}{4}(1+\xi+\eta+\xi\eta) + \frac{1}{4}(1-\xi+\eta-\xi\eta) - 4\cancel{\left[\frac{1}{4}(1-\xi^2)(1-\eta^2)\right]} \ldots$$

$$\ldots + \cancel{(1-\xi^2)(1-\eta^2)} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

$$\boxed{N_1 + N_2 + N_3 + N_4 + N_5 = 1}$$

Problem 5.3

5.3.1) The 8-node hexahedron

$n_e = 6$

$n_f = 8 \times 3 = 24$ Dof

$n_r = 6$

$r = n_f - n_r = 24 - 6 = 18$

$6 \cdot n_g \geq 18$

$n_g \geq 3$

$\therefore$ a gauss rule of $2 \times 2 \times 2$ is required

5.3.2) The 20-node hexahedron

$n_e = 6$

$n_f = 20 \times 3 = 60$ Dof

$n_r = 6$

$r = n_f - n_r = 60 - 6 = 54$

$6 \cdot n_g \geq 54$

$n_g \geq 9$

$\therefore$ a gauss rule of $3 \times 3 \times 3$ is required

5.3.3) The 27-node hexahedron

$n_e = 6$

$n_f = 27 \cdot 3 = 81$ DoF

$n_r = 6$

$r = n_f - n_r = 81 - 6 = 75$

$6 \cdot n_g \geq 75$

$n_g \geq 12.5$

$\boxed{\therefore \text{ a gauss rule of } 3 \times 3 \times 3 \text{ is required}}$

5.3.4) The 60-node hexahedron

$n_e = 6$

$n_f = 60 \cdot 3 = 180$

$n_r = 6$

$r = n_f - n_r = 180 - 6 = 174$

$6 \cdot n_g \geq 174$

$n_g \geq 29$

$\boxed{\therefore \text{ a gauss rule of } 4 \times 4 \times 4 \text{ is required}}$