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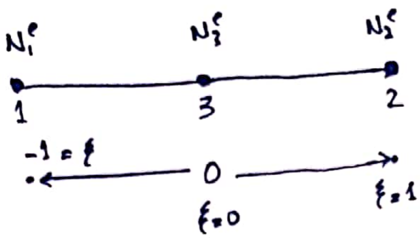
Assignment No. 5.1 CSMD

Isoparametric Straight-mode Bar Element with local System

$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & u_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix}$$



where $\bar{x}_1 = 0$

$\bar{x}_2 = L$

$\bar{x}_3 = \frac{1}{2}L + \alpha L$

Find α_{min} for which $J = \frac{d\bar{x}}{d\xi} = 0$

As $\bar{x} = \sum \bar{x}_i N_i^e = \bar{x}_1 N_1 + \bar{x}_2 N_2 + \bar{x}_3 N_3$

where $N_1 = -\frac{1}{2}\xi(1-\xi)$, $N_2 = \frac{1}{2}\xi(1+\xi)$, $N_3 = (1-\xi^2)$

$$\bar{x} = (0)\left(-\frac{1}{2}\xi(1-\xi)\right) + (L)\left(\frac{1}{2}(\xi(1+\xi)\right) + \left(\frac{1}{2}L+\alpha L\right)(1-\xi^2)$$

As $J = \frac{d\bar{x}}{d\xi} = \bar{x}_1 \frac{dN_1}{d\xi} + \bar{x}_2 \frac{dN_2}{d\xi} + \bar{x}_3 \frac{dN_3}{d\xi}$

$$\frac{d\bar{x}}{d\xi} = (0)\frac{d}{d\xi}\left(-\frac{1}{2}\xi + \frac{1}{2}\xi^2\right) + (L)\frac{d}{d\xi}\left(\frac{1}{2}\xi + \frac{1}{2}\xi^2\right) + \left(\frac{1}{2}L + \alpha L\right)\frac{d}{d\xi}(1-\xi^2)$$

$$\frac{d\bar{x}}{d\xi} = L\left(\frac{1}{2} + \xi\right) + \left(\frac{1}{2}L + \alpha L\right)(-2\xi)$$

$$\frac{d\bar{x}}{d\xi} = L\left(\frac{1}{2} - 2\alpha\xi\right) \quad , \quad \text{For } \frac{d\bar{x}}{d\xi} = 0$$

$$0 = L\left(\frac{1}{2} - 2\alpha\xi\right)$$

So, $\frac{1}{2} - 2\alpha\xi = 0$

For $\xi = -1 \Rightarrow \alpha = -\frac{1}{4}$

$\xi = 1 \Rightarrow \alpha = \frac{1}{4}$

So, hence it is evident at quarter of the element, $J = 0$.

For Strain, $\varepsilon = \frac{d\bar{u}}{dx}$

$$\varepsilon = \frac{d\bar{u}}{dx} = \frac{\partial\bar{u}}{\partial\xi}\cdot\frac{\partial\xi}{\partial x} = \frac{\partial\bar{u}}{\partial\xi}\cdot\frac{1}{J} \qquad \because J = L\left(\frac{1}{2} - 2\alpha\xi\right)$$

As $\bar{u} = \sum \bar{u}_i N_i^e = u_1N_1 + u_2N_2 + u_3N_3$

$$\frac{\partial\bar{u}}{\partial\xi} = u_1\frac{\partial N_1}{\partial\xi} + u_2\frac{\partial N_2}{\partial\xi} + u_3\frac{\partial N_3}{\partial\xi}$$

$$\frac{\partial\bar{u}}{\partial\xi} = u_1\left(-\frac{1}{2}+\xi\right) + u_2\left(\frac{1}{2}+\xi\right) + u_3(-2\xi)$$

So $$\varepsilon = \frac{d\bar{u}}{dx} = \frac{1}{L\left(\frac{1}{2}-2\alpha\xi\right)}\left[u_1\left(-\frac{1}{2}+\xi\right) + u_2\left(\frac{1}{2}+\xi\right) + u_3(-2\xi)\right]$$

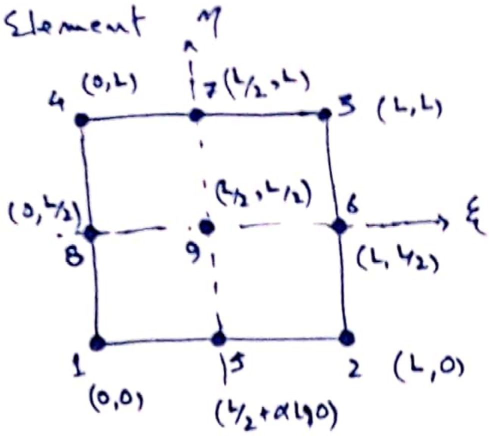
For $\xi = -1$ & $\alpha = -\frac{1}{4} \Rightarrow J = 0$ & $\varepsilon \to \infty$

For $\xi = 1$ & $\alpha = \frac{1}{4} \Rightarrow J = 0$ & $\varepsilon \to \infty$

Hence, it is proved that strain approaches to infinity on both ends of the bar element.

Assignment No. 5.2

9-node plane Stress Element



The shape functions for 9-nodes are followings.

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)\xi\eta \qquad N_5 = -\frac{1}{2}(1-\xi^2)(1-\eta)\eta$$

$$N_2 = -\frac{1}{4}(1+\xi)(1-\eta)\xi\eta \qquad N_6 = \frac{1}{2}(1+\xi)(1-\eta^2)\xi$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)\xi\eta \qquad N_7 = \frac{1}{2}(1-\xi^2)(1+\eta)\eta$$

$$N_4 = -\frac{1}{4}(1-\xi)(1+\eta)\xi\eta \qquad N_8 = -\frac{1}{2}(1-\xi)(1+\eta^2)\xi$$

$$N_9 = (1-\xi^2)(1-\eta^2)$$

And x-y coordinates are.

$x_1 = 0$, $x_2 = L$, $x_3 = L$, $x_4 = 0$, $x_5 = \frac{L}{2} + \alpha L$

$x_6 = L$, $x_7 = \frac{L}{2}$, $x_8 = 0$, $x_9 = \frac{L}{2}$

&

$y_1 = 0$, $y_2 = 0$, $y_3 = L$, $y_4 = L$, $y_5 = 0$

$y_6 = L/2$, $y_7 = L$, $y_8 = L/2$, $y_9 = L/2$

$$\frac{dN_1}{d\xi} = \frac{\eta}{4}(1-2\xi-\eta+2\xi\eta) \qquad \frac{dN_1}{d\eta} = \frac{\xi}{4}(1-2\eta-\xi+2\xi\eta)$$

$$\frac{dN_2}{d\xi} = -\frac{\eta}{4}(1-\eta+2\xi-2\xi\eta) \qquad \frac{dN_2}{d\eta} = -\frac{\xi}{4}(1-2\xi+\xi-2\xi\eta)$$

$$\frac{dN_3}{d\xi} = \frac{\eta}{4}(1+2\xi+\eta+2\xi\eta) \qquad \frac{dN_3}{d\eta} = \frac{\xi}{4}(1+2\eta+\xi+2\xi\eta)$$

$$\frac{dN_4}{d\xi} = -\frac{\eta}{4}(1-2\xi+\eta-2\xi\eta) \qquad \frac{dN_4}{d\eta} = -\frac{\xi}{4}(1-\xi+2\eta-2\xi\eta)$$

$$\frac{dN_5}{d\xi} = \xi\eta(1-\eta) \qquad \frac{dN_5}{d\xi} = -\frac{1}{2}(1-\xi^2-2\eta+2\eta\xi^2)$$

$$\frac{dN_6}{d\xi} = \frac{1}{2}(1+\xi-\eta^2-2\xi\eta^2) \qquad \frac{dN_6}{d\eta} = -\xi\eta(1+\xi)$$

$$\frac{dN_7}{d\xi} = -\xi\eta(1+\eta) \qquad \frac{dN_7}{d\eta} = \frac{1}{2}(1-\xi^2+2\eta-2\eta\xi^2)$$

$$\frac{dN_8}{d\xi} = -\frac{1}{2}(1-2\xi-\eta^2+2\xi\eta^2) \qquad \frac{dN_8}{d\eta} = \xi\eta(1-\xi)$$

$$\frac{dN_9}{d\xi} = 2\xi(\eta^2-1) \qquad \frac{dN_9}{d\eta} = 2\eta(\xi^2-1)$$

And we know Jacobian is

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \quad \text{where} \quad \frac{\partial x}{\partial \xi} = \sum_{i=1}^{9} \bar{x}_i \frac{\partial N_i}{\partial \xi}, \quad \frac{\partial x}{\partial \eta} = \sum_{i=1}^{9} \bar{x}_i \frac{\partial N_i}{\partial \eta}$$

$$\frac{\partial y}{\partial \xi} = \sum_{i=1}^{9} \bar{y}_i \frac{\partial N_i}{\partial \eta}, \quad \frac{\partial y}{\partial \eta} = \sum_{i=1}^{9} \bar{y}_i \frac{\partial N_i}{\partial \eta}$$

$$\frac{\partial x}{\partial \xi} = \sum_{i=1}^{9} \bar{x}_i \frac{\partial N_i}{\partial \xi} = x_1 \frac{\partial N_1}{\partial \xi} + x_2 \frac{\partial N_2}{\partial \xi} + x_3 \frac{\partial N_3}{\partial \xi} + x_4 \frac{\partial N_4}{\partial \xi}$$
$$+ x_5 \frac{\partial N_5}{\partial \xi} + x_6 \frac{\partial N_6}{\partial \xi} + x_7 \frac{\partial N_7}{\partial \xi} + x_8 \frac{\partial N_8}{\partial \xi} + x_9 \frac{\partial N_9}{\partial \xi}$$

$$\frac{\partial x}{\partial \xi} = (0)\left[\frac{\eta}{2}(1-2\xi-\eta+2\xi\eta)\right] + (L)\left[-\frac{\eta}{4}(1-\eta+2\xi-2\xi\eta)\right] +$$
$$+ (L)\left[\frac{\eta}{4}(1+2\xi+\eta+2\xi\eta)\right] + (0)\left[-\frac{\eta}{4}(1-2\xi+\eta-2\xi\eta)\right] +$$
$$+ \left(\frac{L}{2}+\alpha L\right)\left[\xi\eta(1-\eta)\right] + (L)\left[\frac{1}{2}(1+\xi-\eta^2-2\xi\eta^2)\right] + \left(\frac{L}{2}\right)\left[-\xi\eta(1+\eta)\right]$$
$$+ (0)\left[-\frac{1}{2}(1-2\xi-\eta^2+2\xi\eta^2)\right] + \left(\frac{L}{2}\right)\left[2\xi(\eta^2-1)\right]$$

$$\frac{\partial x}{\partial \xi} = \alpha L \xi \eta (1-\eta) + \frac{L}{2}$$ (5)

Similarly,

$$\frac{\partial x}{\partial \eta} = \sum_{i=1}^{9} x_1 \frac{\partial N_1}{\partial \eta} + x_2 \frac{\partial N_2}{\partial \eta} + x_3 \frac{\partial N_3}{\partial \eta} + x_4 \frac{\partial N_4}{\partial \eta} + x_5 \frac{\partial N_5}{\partial \eta} + x_6 \frac{\partial N_6}{\partial \eta}$$

$$+ x_7 \frac{\partial N_7}{\partial \eta} + x_8 \frac{\partial N_8}{\partial \eta} + x_9 \frac{\partial N_9}{\partial \eta}$$

$$\frac{\partial x}{\partial \eta} = \alpha L \left[-\frac{1}{2} + \eta + \frac{\xi^2}{2} - \eta \xi^2 \right]$$

$$\frac{\partial y}{\partial \xi} = \sum_{i=1}^{9} y_1 \frac{\partial N_1}{\partial \xi} + y_2 \frac{\partial N_2}{\partial \xi} + y_3 \frac{\partial N_3}{\partial \xi} + y_4 \frac{\partial N_4}{\partial \xi} + y_5 \frac{\partial N_5}{\partial \xi} + y_6 \frac{\partial N_6}{\partial \xi}$$

$$+ y_7 \frac{\partial N_7}{\partial \xi} + y_8 \frac{\partial N_8}{\partial \xi} + y_9 \frac{\partial N_9}{\partial \xi}$$

$$\frac{\partial y}{\partial \xi} = 0$$

$$\frac{\partial y}{\partial \eta} = \sum_{i=1}^{9} y_1 \frac{\partial N_1}{\partial \eta} + y_2 \frac{\partial N_2}{\partial \eta} + y_3 \frac{\partial N_3}{\partial \eta} + y_4 \frac{\partial N_4}{\partial \eta} + y_5 \frac{\partial N_5}{\partial \eta} + y_6 \frac{\partial N_6}{\partial \eta}$$

$$+ y_7 \frac{\partial N_7}{\partial \eta} + y_8 \frac{\partial N_8}{\partial \eta} + y_9 \frac{\partial N_9}{\partial \eta}$$

$$\frac{\partial y}{\partial \eta} = \frac{L}{2}$$, So

$$J = \begin{bmatrix} \alpha L \xi \eta (1-\eta) + \frac{L}{2} & 0 \\ \alpha L \left(-\frac{1}{2} + \eta + \frac{\xi^2}{2} - \eta \xi^2\right) & \frac{L}{2} \end{bmatrix} \Rightarrow |J| = \frac{L}{2}\left(\alpha L \xi \eta (1-\eta) + \frac{L}{2}\right)$$

For $|J| = 0$ and end pt. $\eta = 1$, $\xi = 1$

$$\frac{L}{2}\left(\alpha L \xi \eta (1-\eta) + \frac{L}{2}\right) = 0 \Rightarrow \alpha = \frac{1}{4}$$

$\therefore$ At quarter of the element.