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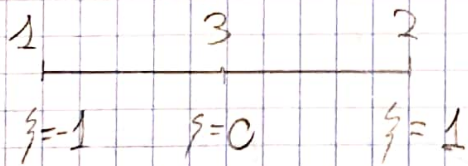
Assignment 5.1

$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x_1} & \bar{x_2} & \bar{x_3} \\ \bar{u_1} & \bar{u_2} & \bar{u_3} \end{bmatrix} \begin{bmatrix} N_1(\xi) \\ N_2(\xi) \\ N_3(\xi) \end{bmatrix}$$

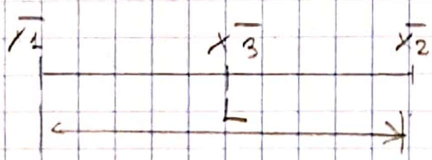
$\bar{x_1} = 0$, $\bar{x_2} = L$, $\bar{x_3} = \frac{1}{2} l + \alpha l$

- Show that $\alpha = \pm 1/4$ for $J = d\bar{x}/d\bar{\xi}$ vanishes. Interpret singularity showing axial strain becomes infinite at end point

- 1D quadratic element:



Cartesian coordinates.



$x_1 = 0$

$x_2 = L$

$x_3 = \frac{1}{2} l + \alpha l$

Jacobian

$J = \frac{dx}{d\xi}$, Jacobian calculated for $J = 0$ to obtain values for α.

* First, it is required to calculate $\bar{x}$

$$\bar{x} = \bar{x}_1 N_1(\xi) + \bar{x}_2 N_2(\xi) + \bar{x}_3 N_3(\xi)$$

* Shape functions:

$$N = a + b\xi + c\xi^2 \qquad \Big\} \xi: \text{ isoparametric coordinate}$$

Substituting the values of ξ for each point:

$$N_1^e = \frac{1}{2}\xi(\xi - 1)$$

$$N_2^e = \frac{1}{2}\xi(1 + \xi)$$

$$N_3^e = 1 - \xi^2$$

* Introducing those in the definition of $\bar{x}$ with the cartesian coordinates $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$:

$$\bar{x} = \left[\frac{1}{2}\xi(\xi-1)\right]\cdot 0 + \left[\frac{1}{2}\xi(1+\xi)\right]L + \left[1-\xi^2\right]\left(\frac{1}{2}L + \alpha L\right) =$$

$$= \frac{1}{2}\xi(1+\xi)L + \frac{1}{2}L(1-\xi^2) + \alpha L(1-\xi^2)$$

*

$$J = \frac{d\bar{x}}{d\xi} = \left(\frac{1}{2} + \frac{1}{2}\cdot 2\xi\right)L + \left(\frac{-2\xi}{2}L\right) +$$

$$+ (-2\xi\alpha L) = \frac{L}{2} - 2\xi\alpha L$$

For J=0

$$J=0 \Rightarrow 0 = \frac{L}{2} - 2\xi\alpha L$$

$$-2\xi\alpha = -\frac{1}{2} \; ; \; \boxed{\alpha = \frac{1}{4\xi}}$$

The maximum values for ξ are 1 and -1, so:

$$\xi = 1 \Rightarrow \alpha = \frac{1}{4}$$

$$\xi = -1 \Rightarrow \alpha = \frac{-1}{4}$$

So, the minimum values for α which Jacobian vanishes are:

$$\boxed{\alpha = \pm \frac{1}{4}}$$

Interpreting the result as a singularity:

Strain matrix is:

$$[\varepsilon] = [B]\,[J^{-1}]\,[u]$$

$$[J^{-1}] = \frac{d\xi}{d\bar{x}}$$

+ Previously, it is showed that at the end points, the Jacobian is equal to 0.

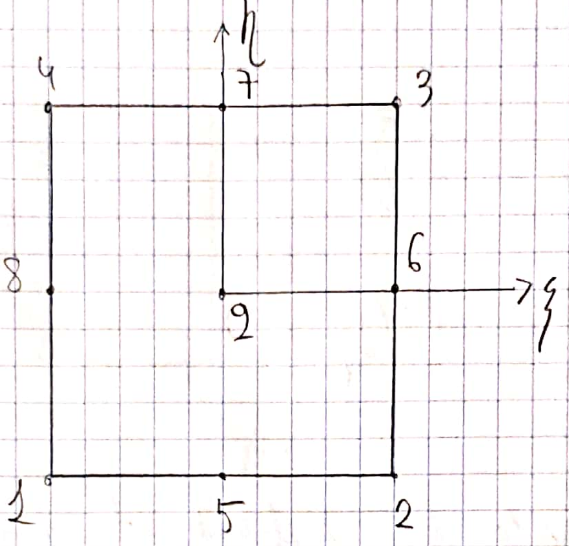
$$J = \frac{d\bar{x}}{d\xi} = 0 \Rightarrow J^{-1} = \frac{1}{J} = \frac{1}{0} = \infty$$

So, as $J^{-1} = \infty$:

$$\varepsilon = B J^{-1} u \rightarrow \boxed{\varepsilon = \infty}$$

Assignment 5.2

Extend for 9-noble plane stress element.
Move node 5 towards 2 until the Jacobian determinant at 2 vanishes



$\eta \in [-1, 1]$

$\xi \in [-1, 1]$

The position of node 5 is $(\bar{x}, \bar{y}) = (0, -l/2)$ and it moves tangetially to node 2, so: $(\bar{x}, \bar{y}) = (\alpha l, -l/2)$

+ Shape functions:

$$N_1 = \frac{1}{4} \xi \eta (\xi - 1)(\eta - 1) \qquad N_5 = \frac{1}{2} \eta (1 - \xi^2)(\eta - 1)$$

$$N_2 = \frac{1}{4} \xi \eta (\xi + 1)(\eta - 1) \qquad N_6 = \frac{1}{2} \xi (\xi + 1)(1 - \eta^2)$$

$$N_3 = \frac{1}{4} \xi \eta (\xi + 1)(\eta + 1) \qquad N_7 = \frac{1}{2} \eta (1 - \xi^2)(\eta + 1)$$

$$N_4 = \frac{1}{4} \xi \eta (\xi - 1)(\eta + 1) \qquad N_8 = \frac{1}{2} \xi (\xi - 1)(1 - \eta^2)$$

$$N_9 = (1 - \xi^2)(1 - \eta^2)$$

* As before, the relation between the cartesian and isoparametric coordinates with the help of the shape functions:

$$\bar{x} = \sum_{i=1}^{9} \bar{x}_i \, N_i(\xi, \eta) \qquad y = \sum_{i=1}^{9} y_i \, N_i(\xi, \eta)$$

* Jacobian matrix:

$$\underline{J} = \begin{bmatrix} \partial\bar{x}/\partial\xi & \partial\bar{y}/\partial\xi \\ \partial\bar{x}/\partial\eta & \partial\bar{y}/\partial\eta \end{bmatrix}$$

* This Jacobian has to be calculated at node 2 and it is equal to 0.

Node 2 $\Rightarrow$ $\xi = 1$, $\eta = -1$

* All the derivatives has been done with help of the software Matlab: The sums to get $\bar{x}$ and $\bar{y}$ as well, yielding to:

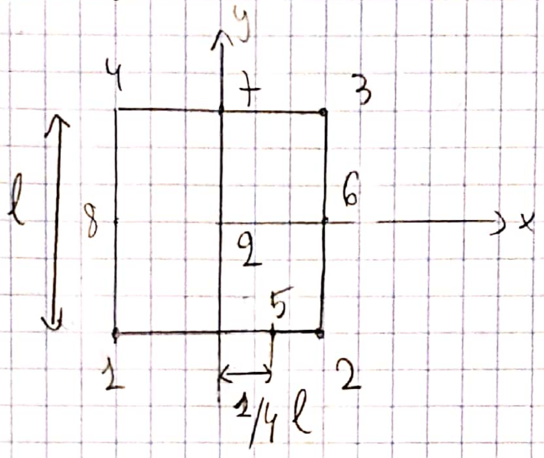
$$J_{node\ 2} = \begin{bmatrix} l/2 - 2\alpha l & 0 \\ 0 & l/2 \end{bmatrix}$$

So, $|J| = \frac{l^2}{4} - l^2\alpha = 0$; $\boxed{\alpha = \frac{1}{4}}$

Scanned with CamScanner

So, again when $\alpha = \frac{1}{4}$ it's a singular problem, where the strain tend to infinity and fracture mechanics plays a role.

Final position of node 5 : $(x, y) = \left(\frac{1}{4} l \, , -1/2\right)$



These results obtained (for $|J| = 0$) conclude that for 2D elements with high order the location of the coordinates (1, 2, 3 and 4) is not enough.
The coordinates that are not placed on the corners, such as node 5, have to be placed close to the cartesian coordinates in order to avoid local distorsions.
This have an application for cracks in construction (linear fracture mechanics).