Computational Structural Mechanics and Dynamics

Assignment 4 - Isoparametric representation and Structures of revolution

Federico Parisi
federicoparisi95@gmail.com

Universitat Politècnica de Catalunya — March 4th, 2020

Assignment 4.1:

A 3-node straight bar element is defined by 3 nodes: 1, 2 and 3 with axial coordinates x_1, x_2 and x_3 respectively as illustrated in figure below. The element has axial rigidity EA, and length $l = x_{2-x_1}$. The axial displacement is u(x). The 3 degrees of freedom are the axial node displacement u_1, u_2 and u_3. The isoparametric definition of the element is

$$\begin{bmatrix} 1 \\ x \\ u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix} \tag{1}$$

in which $N_1^e(\xi)$ are the shape functions of a three bar element. Node 3 lies between 1 and 2 but is not necessarily at the midpoint x = l/2. For convenience define,

$$x_1 = 0,\ x_2 = l,\ x_3 = (\frac{1}{2} - \alpha)l \tag{2}$$

where $-\frac{1}{2} < \alpha < \frac{1}{2}$ characterizes the location of node 3 with respect to the element center. If $\alpha = 0$ node 3 is located at the midpoint between 1 and 2.

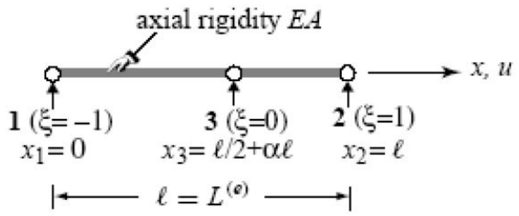

Figure 1: The three-node bar element in its local system

.

1. From (1) and the second equation of (2) get the Jacobian $J = \frac{dx}{d\xi}$ in terms of l, α and ξ. Show that

- if – ¼ < α < ¼ then J > 0 over the whole element $-1 \leq \xi \leq 1$.
- if $\alpha = 0, J = \frac{1}{2}$ is a constant over the element.

2. Obtain the 1x3 strain displacement matrix B relating $e = du/dx = Bu^e$ where ue is the column 3-vector of the node displacement u1, u2 and u3. The entries of B are functions of l, α and ξ.

Answer 1

The element that is going to be analyzed is a 3-node bar. So the element, defined by three nodes, is a quadratic linear element. This means that in order to describe it, three parabolic shape functions are needed. As always, the shape functions have to satisfy $1 = \sum N_i$, easy to see if it is computed the 1^{st} row of system (1). The general way of obtaining the shape function, for n number of nodes is:

$$N_j = \prod_{i=1, i\neq j}^{n} \frac{(\xi_i - \xi)}{(\xi_i - \xi_j)} \tag{3}$$

that in this case will return to us three functions:

$$\begin{aligned} N_1 &= \frac{\xi}{2}(\xi - 1) \\ N_2 &= \frac{\xi}{2}(\xi + 1) \\ N_3 &= 1 - \xi^2 \end{aligned} \tag{4}$$

As can be noticed, these three functions are defined in the isoparametric coordinates and they fulfill the requirements, going to 1 in the respective node and 0 in the others two.

In order to represent those in the cartesian system, it is needed to find the Jacobian. Finding the Jacobian means to compute $\frac{d\mathbf{x}}{d\xi}$ that is exactly the value of the Jacobian. Being in 1-D will make the Jacobian a scalar, but in 2-D will be a matrix 2X2 etc...

x is defined, from the second row of (1) as:

$$\mathbf{x} = \sum_{i=1}^{3} x_i N_i \tag{5}$$

in which x_i are the cartesian coordinates: $x_1 = 0,\ x_2 = l,\ x_3 = l(\frac{1}{2} + \alpha)$ in this case, and N_i are the shape functions defined in (4).

$$\mathbf{x} = l\frac{\xi}{2}(1+\xi) + l(\frac{1}{2} + \alpha)(1 - \xi^2) \tag{6}$$

Deriving it respect to ξ, we obtain the Jacobian that, as said before, is a scalar:

$$J = \frac{d\mathbf{x}}{d\xi} = \frac{l}{2} - 2\alpha l\xi = l(\frac{1}{2} - 2\alpha\xi) \tag{7}$$

As can be noticed, the resultant Jacobian is a function of α and ξ. This could have been easily predicted as the mid point is a function of α, and the isoparametric coordinate appears when the mid point won't be equidistant from the two external nodes.

- Taking the equation (7), the sign can be studied in order to define a value for α:

$$\begin{aligned} J &> 0 \\ \frac{1}{2} - 2\alpha\xi &> 0 \end{aligned} \tag{8}$$

  Studying it for the two boundary cases ($\xi = 1$ and $\xi = -1$)

$$\begin{aligned} \xi = 1 &\longrightarrow \frac{1}{2} - 2\alpha > 0 \longrightarrow \alpha < \frac{1}{4} \\ \xi = -1 &\longrightarrow \frac{1}{2} + 2\alpha > 0 \longrightarrow \alpha > -\frac{1}{4} \end{aligned} \tag{9}$$

  so the Jacobian will be positive for $-\frac{1}{4} < \alpha < \frac{1}{4}$.

- As said above, if the $3^r d$ node is exactly in the middle of the element, the Jacobian won't depend from ξ. This because the equi-distance between the two external nodes with the mid one makes the Jacobian (that is the coordinates transformation) constant everywhere. It can be seen simply substituting in (7) the value $\alpha = 0$ and the Jacobian takes the form of:

$$J = \frac{1}{2} \tag{10}$$

  that does not depend on ξ so it is constant over the element as predicted.

Answer 2

The matrix B that defines $e = Bu^e$ for the element, is defined as the derivative of the shape functions respect to x. Using the isoparametric representation and remembering that $J = \frac{dx}{d\xi}$, the derivative respect to x is written as:

$$B_i = \frac{dN_i}{dx} = \frac{d\xi}{dx}\frac{dN_i}{d\xi} = J^{-1}\frac{dN_i}{d\xi} \tag{11}$$

and it results in the following 1x3 matrix:

$$\begin{aligned} J^{-1} &= [l(\frac{1}{2} - 2\alpha\xi)]^{-1} \\ B &= J^{-1}\begin{bmatrix} \xi - \frac{1}{2} & \xi + \frac{1}{2} & -2\xi \end{bmatrix} \end{aligned} \tag{12}$$

Substituting the nodes coordinates in the Jacobian, is obtained the matrix:

$$B = \begin{bmatrix} \frac{2\xi - 1}{l(1+4\alpha)} & \frac{2\xi+1}{l(1-4\alpha)} & -\frac{4\xi}{l} \end{bmatrix} \tag{13}$$

The component of the B matrix related to the 3^{rd} node is independent from α. That's because that node is the center of the isoparametric system ($\xi = 0$).

To finish the 1^{st} part of the assignment, some considerations have to be done. In this case it wasn't necessary to use the isoparametric representation as the element was straight and not a complex one. Using the isoparametric representation is useful when the elemnt is curved and it would be very difficult using the cartesian coordinates. In this particular case, using the isoparameric ones, reduced the complexity due to the changing of the position of the 3^{rd} node.

Assignment 4.2

1. Compute the entries of Ke for the following axisymmetric triangle:

$$r_1 = 0,\ r_2 = r_3 = a,\ z_1 = z_2 = 0,\ z_3 = b$$

The material is isotropic with = 0 for which the stress-strain matrix is,

$$\mathbf{E} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{14}$$

2. Show that the sum of the rows (and columns) 2, 4 and 6 of Ke must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.

3. Compute the consistent force vector f^e for gravity forces b = [0, –g]T.

Answer 1

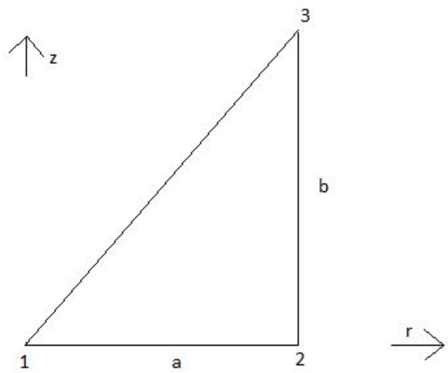

Figure 2: Axisymmetric triangle

Working with an axisymmetric triangle means working with polar coordinates (r, z and θ) taking into account that is uniform over θ. The stiffness matrix is defined by:

$$K = \int_V B^T DBd(vol) \tag{15}$$

According to that relationship and considering we are in axisymmetric case, the volume integral over the whole ring can be written as following:

$$K^e_{ij} = 2\pi \int B_i^T DB_j rdrdz \tag{16}$$

To compute the stiffness matrix it is needed to compute the matrix B and D. Due to the nature of the material, is an isotropic material with $\nu = 0$, the resultant D matrix coincides with the E one given at the beginning of the assignment (14). It can be noticed as the general definition of the D matrix for an isotropic material is:

$$D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ & 1-\nu & \nu & 0 \\ & & 1-\nu & 0 \\ sym & & & (1-2\nu)/2 \end{bmatrix} \tag{17}$$

and for $\nu = 0$ it takes the form of (14):

$$D = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ sym & & & \frac{1}{2} \end{bmatrix} \tag{18}$$

The B matrix is defined as always:

$$B_i = \begin{bmatrix} \frac{\partial N_i}{\partial r} & 0 \\ 0 & \frac{\partial N_i}{\partial z} \\ \frac{N_i}{r} & 0 \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial r} \end{bmatrix}, \ i = 1, 2, 3 \tag{19}$$

And the shape function, for the node i is defined as following

$$\begin{aligned} N_i &= \frac{a_i + b_i r + c_i z}{2A} \\ a_i &= r_j z_m - r_m z_j \\ b_i &= z_j - z_m \\ c_i &= r_m - r_j \end{aligned} \tag{20}$$

Computing the shape functions for each node, taking into account $A = ab/2$ the resultant N-vector is:

$$N = \frac{1}{ab} \begin{bmatrix} ab - br & br - az & az \end{bmatrix} \tag{21}$$

in which the properties of the shape functions are satisfied, taking the value one in the respective node and zero in the others. Computing the B matrix:

$$B = \frac{1}{ab} \begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ \frac{ab}{r} - b & 0 & b - \frac{az}{r} & 0 & \frac{az}{r} & 0 \\ 0 & -b & -a & b & a & 0 \end{bmatrix} \tag{22}$$

Remembering the definition of K given in equation (16), it has to be integrated in order to write it. The integration to be computed is not as easy as in the plane stress problem as the B matrix depends on the coordinates, so it can be done in two ways. The first one is integrating term by term, the other one is doing by numerical integration. The simplest way to do so is evaluating all the quantities for a centroidal points, defined as the mean point:

$$\begin{aligned} \mathbf{r} &= \frac{1}{3} \sum_{i=1}^{3} r_i = \frac{2a}{3} \\ \mathbf{z} &= \frac{1}{3} \sum_{i=1}^{3} z_i = \frac{b}{3} \end{aligned} \tag{23}$$

So my stiffness matrix to be computed will be so similar to the one in (16), but with the numerical integration applied:

$$K = 2\pi B^T D B \mathbf{r} A \tag{24}$$

Substituting everything in the previous formula and computing the B matrix with the r and z coordinates compute in (23), we obtain the global stiffness matrix:

$$K = \frac{E\pi}{ab} \begin{bmatrix} \frac{5b^2}{4} & 0 & -\frac{3b^2}{4} & 0 & \frac{b^2}{4} & 0 \\ 0 & \frac{b^2}{2} & \frac{ab}{2} & -\frac{b^2}{2} & -\frac{ab}{2} & 0 \\ -\frac{3b^2}{4} & \frac{ab}{2} & \frac{5b^2+2a^2}{4} & -\frac{ab}{2} & \frac{b^2-2a^2}{4} & 0 \\ 0 & -\frac{b^2}{2} & -\frac{ab}{2} & \frac{2a^2+b^2}{2} & \frac{ab}{2} & -a^2 \\ \frac{b^2}{4} & -\frac{ab}{2} & \frac{b^2-2a^2}{4} & \frac{ab}{2} & \frac{b^2+2a^2}{4} & 0 \\ 0 & 0 & 0 & -a^2 & 0 & a^2 \end{bmatrix} \tag{25}$$

As expected the matrix is symmetric.

Answer 2

As can be noticed, the sum of the rows 2, 4 and 6 of the stiffness matrix, vanish. Due to the symmetry of the matrix, also the respective columns will vanish as well. This is because the equilibrium in the z-direction has to be guaranteed by the stiffness matrix as there are not any constraints given by the problem.

On the other hand, the equilibrium on the r-direction is guaranteed by the symmetry of the problem along the z-axis. That explains why the rows (and columns) related to the r-coordinates of each node are different than zero.

Answer 3

To compute the forces vector, considering only the body forces, they are computed in this way:

$$f = -2\pi \int_S N b r dr dz \tag{26}$$

Considering the centroidal coordinate as done while computing the stiffness matrix:

$$f = -2\pi N(\mathbf{r}, \mathbf{z}) \begin{bmatrix} b_r \\ b_z \end{bmatrix} \mathbf{r} A \tag{27}$$

in which $\mathbf{r}$ and $\mathbf{z}$ are defined at (23). Considering the body forces constant:

$$f = -\frac{2}{3}\pi a \begin{bmatrix} \frac{ab}{3} & 0 \\ 0 & \frac{ab}{3} \\ \frac{ab}{3} & 0 \\ 0 & \frac{ab}{3} \\ \frac{ab}{3} & 0 \\ 0 & \frac{ab}{3} \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix} \tag{28}$$

computing it

$$f = -\frac{2}{9}\pi a^2 b g \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tag{29}$$

It makes sense as the body forces are acted only along the vertical direction.