Computational Structural Mechanics and Dynamics
Assignment 4

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1 Problem Description

1.1 Assignment 4.1

A 3-node bar element is defined by 3 nodes: 1, 2 and 3 with axial coordinates x_1, x_2 and x_3 respectively, as illustrated in the figure below. The element has axial rigidity EA, and length $l = x_1 - x_2$. The axial displacement is u(x). The 3 degrees of freedom are the axial node displacements u_1, u_2 and u_3. The isoparametric definition of the element is:

$$\begin{bmatrix} 1 \\ x \\ u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix} \tag{1}$$

in which $N_i^e(\xi)$ are the shape functions of a 3-node bar element. Node 3 lies between 1 and 2 but it is not necessarily at the midpoint x = 1/2. For convenience, define:

$$x_1 = 0 \quad x_2 = l \quad x_3 = (\frac{1}{2} + \alpha)l \tag{2}$$

where $-\frac{1}{2} < \alpha < \frac{1}{2}$ characterizes the location of node 3 with respect to the element center. If $\alpha = 0$ node 3 is located at the midpoint between nodes 1 and 2.

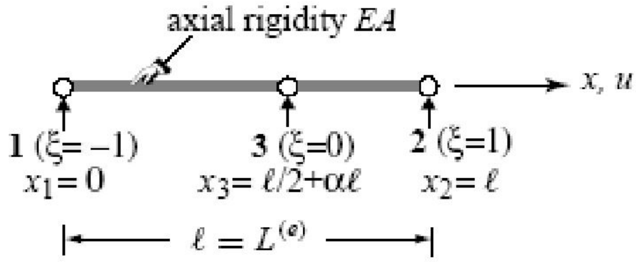

Figure 1: 3-node bar element in local system

1. From (2) and the second equation of (1) get the Jacobian $J = \frac{dx}{d\xi}$ in terms of l, α and ξ. Show that,
   - if $-\frac{1}{4} < \alpha < \frac{1}{4}$, $J > 0$ over the whole element $-1 < \xi < 1$.
   - if $\alpha = 0$, $J = l/2$ is a constant over the element.
2. Obtain the strain displacement matrix B relating $e = \frac{du}{dx} = \mathbf{B}\mathbf{u^e}$ where $\mathbf{u^e}$ is the column 3-vector of the node displacement u_1, u_2 and u_3. The entries of $\mathbf{B}$ are functions of l, α and ξ.

1.2 Assignment 4.2

1. Compute the entries of $\mathbf{K^e}$ for the following axisymmetric triangle:

$$r_1 = 0, \qquad r_2 = r_3 = a, \qquad z_1 = z_2 = 0, \qquad z_3 = b \tag{3}$$

The material is isotropic with $v = 0$ for which the stress-strain matrix is:

$$\mathbf{C} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & \frac{1}{2} \end{bmatrix} \tag{4}$$

2. Show that the sum of the rows (and columns) 2, 4 and 6 of $\mathbf{K^e}$ must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.

3. Compute the consistent force vector $\mathbf{f^e}$ for gravity forces $\mathbf{b} = [0, -g]^T$.

2 Solution

2.1 Assignment 4.1

For a 3-node **isoparametric** bar element, the node coordinates are defined as:

$$\xi_1 = -1 \qquad \xi_2 = 1 \qquad \xi_3 = 0 \tag{5}$$

Therefore, the shape functions take the form:

$$N_1(\xi) = \frac{1}{2}\xi(\xi - 1) \qquad N_2(\xi) = \frac{1}{2}\xi(\xi + 1) \qquad N_3(\xi) = 1 - \xi^2 \tag{6}$$

and the corresponding derivatives:

$$\frac{dN_1}{\xi} = \xi - \frac{1}{2} \qquad \frac{dN_2}{\xi} = \xi + \frac{1}{2} \qquad \frac{dN_3}{\xi} = -2\xi \tag{7}$$

Now, from equation (1) we know that x can be expressed as a function of ξ as:

$$x = \sum_{i=1}^{3} x_i N_i(\xi) \tag{8}$$

We may then obtain the Jacobian by deriving x with respect to ξ

$$J = \frac{dx}{d\xi} = \sum_{i=1}^{3} x_i \frac{dN_i(\xi)}{d\xi} \tag{9}$$

$$\boxed{J = l\big(\frac{1}{2} - 2\alpha\xi\big)} \tag{10}$$

Knowing the expression for the Jacobian as a function of α, ξ and l, it is possible to analyze how it is affected for different values of α.

- $\boldsymbol{-\frac{1}{4} < \alpha < \frac{1}{4}}$

  It is known that $l > 0$ and $-1 < \xi < 1$. Therefore, from equation (10) we may deduce that for the Jacobian to be positive ($J > 0$), the following inequality must be fulfilled:

  $$2\alpha\xi < \frac{1}{2} \tag{11}$$

  In order for this expression to be true for any ξ between -1 and 1, α must be:

  $$|a| < \frac{1}{4} \quad \rightarrow \quad \boxed{-\frac{1}{4} < \alpha < \frac{1}{4}} \tag{12}$$

- $\alpha = \mathbf{0}$

  In this case, node 3 is located exactly at the center of the element and the Jacobian becomes a constant:

$$\boxed{J = \frac{l}{2}} \tag{13}$$

Now we are interested in finding the strain displacement matrix **B**, which is expressed as:

$$B = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} & \frac{dN_3}{dx} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{dN_1}{d\xi} & \frac{dN_2}{d\xi} & \frac{dN_3}{d\xi} \end{bmatrix} \tag{14}$$

The derivatives of the shape functions are known from equation (7) and the inverted Jacobian takes the form:

$$J^{-1} = \frac{2}{l(1 - 4\alpha\xi)} \tag{15}$$

Hence, B becomes:

$$\boxed{B = \frac{2}{l(1 - 4\alpha\xi)} \left[\left(\xi - \frac{1}{2} \right) \quad \left(\xi + \frac{1}{2} \right) \quad \left(-2\xi \right) \right]} \tag{16}$$

If the nodal displacements u_i are known, the strains at any point of the element may then be computed as:

$$e = \frac{du}{dx} = \mathbf{B}\mathbf{u}^{\mathbf{e}} = J^{-1} \sum_{i=1}^{3} u_i \frac{dN_i}{d\xi} \tag{17}$$

$$\boxed{e = \frac{2}{l(1 - 4\alpha\xi)} \left[u_1 \left(\xi - \frac{1}{2} \right) + u_2 \left(\xi + \frac{1}{2} \right) + u_3 \left(-2\xi \right) \right]} \tag{18}$$

2.2 Assignment 4.2

The axisymmetric triangular structure described by the given coordinates has the following geometrical configuration:

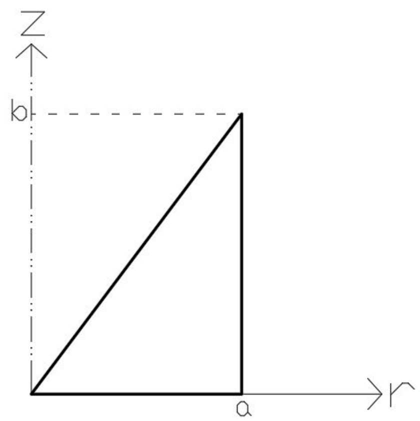

Figure 2: Axisymmetric Structure

The stiffness matrix of an axisymmetric structure is given by:

$$K = \int_\Omega B^T C B d\Omega = 2\pi \int B^T C B r dr dz \tag{19}$$

$$B = DN = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \tag{20}$$

Applying the following linear shape functions:

$$N_1 = 1 - \frac{r}{a} \qquad N_2 = \frac{r}{a} - \frac{z}{b} \qquad N_3 = \frac{z}{b} \tag{21}$$

and their respective derivatives:

$$\frac{\partial N_1}{\partial r} = -\frac{1}{a} \qquad \frac{\partial N_2}{\partial r} = \frac{1}{a} \qquad \frac{\partial N_3}{\partial r} = 0 \tag{22}$$

$$\frac{\partial N_1}{\partial z} = 0 \qquad \frac{\partial N_2}{\partial z} = -\frac{1}{b} \qquad \frac{\partial N_3}{\partial z} = \frac{1}{b} \tag{23}$$

The kinematic matrix B takes the form:

$$B = \begin{bmatrix} -\frac{1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b} & 0 & \frac{1}{b} \\ \left(\frac{1}{r} - \frac{1}{a}\right) & 0 & \left(\frac{1}{a} - \frac{z}{rb}\right) & 0 & \frac{z}{rb} & 0 \\ 0 & -\frac{1}{a} & -\frac{1}{b} & \frac{1}{a} & \frac{1}{b} & 0 \end{bmatrix} \tag{24}$$

However, the stiffness matrix is substantially more complicated to compute than a plane stress or plane strain matrix since the kinematic matrix B depends on the coordinates. According to Zienkiewicz (see reference 2), a simple way to compute the integral numerically is to evaluate all quantities for a centroidal point. Hence, the expression for the stiffness matrix becomes:

$$K = 2\pi \overline{B}^T C \overline{B} \overline{r} \Delta dr dz \tag{25}$$

Where:

- $\overline{r}$ and $\overline{z}$ are the coordinates of the element centroid.
- $\Delta = \frac{ab}{2}$ is the area of the triangle.
- $\overline{B}$ is the kinematic matrix evaluated at the centroidal point.

For this particular case, the coordinates of the centroid are:

$$\overline{r} = \frac{2}{3}a \qquad \overline{z} = \frac{b}{3} \tag{26}$$

Therefore, $\overline{B}$ takes the form:

$$\overline{B} = \begin{bmatrix} -\frac{1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b} & 0 & \frac{1}{b} \\ \frac{1}{2a} & 0 & \frac{1}{2a} & 0 & \frac{1}{2a} & 0 \\ 0 & -\frac{1}{a} & -\frac{1}{b} & \frac{1}{a} & \frac{1}{b} & 0 \end{bmatrix} \tag{27}$$

Then, applying equation (25) we compute K:

$$\boxed{K = \frac{2\pi}{3}E \begin{bmatrix} \frac{5b}{4} & 0 & -\frac{3b}{4} & 0 & \frac{b}{4} & 0 \\ & \frac{b}{2} & \frac{a}{2} & -\frac{b}{2} & -\frac{a}{2} & 0 \\ & & (a^2b)\left(\frac{5}{4a^2}+\frac{1}{2b^2}\right) & -\frac{a}{2} & (a^2b)\left(\frac{1}{4a^2}-\frac{1}{2b^2}\right) & 0 \\ & & & a^2b\left(\frac{1}{2a^2}+\frac{1}{b^2}\right) & \frac{a}{2} & \frac{-a^2}{b} \\ & & & & (a^2b)\left(\frac{1}{4a^2}+\frac{1}{2b^2}\right) & 0 \\ Sym & & & & & \frac{a^2}{b} \end{bmatrix}} \tag{28}$$

It may be observed that the sum of rows or columns 2, 4, and 6 results in a row or column of zeros because of equilibrium. If degree of freedom 2 is being subjected to a unitary displacement by applying the force corresponding to K_{22} on it, degrees of freedom 4 and 6 must be held in place (restrained) by applying the forces corresponding to K_{42} and K_{62} on them, respectively. However, this is not the case for rows or columns 1, 3 and 5, due to the fact that we are dealing with an axisymmetric structure and there will be an element radially mirroring the analyzed one on the other side of the symmetry axis, which will have a stiffness opposing that of this element for radial degrees of freedom. Therefore, if the complete structure was to be analyzed, the sum of the odd rows and columns of the global stiffness matrix would go to zero.

Now, for the body forces vector, an approximation like the one used for the computation of the stiffness matrix may be employed. Assuming that density is constant throughout the domain, according to Zienkiewicz (see reference 2), the body forces of node i can be approximated as:

$$f_i^e = -2\pi \begin{bmatrix} b_r \\ b_z \end{bmatrix} \frac{\overline{r}\Delta}{3} = -\frac{2\pi}{9}a^2b \begin{bmatrix} 0 \\ -g \end{bmatrix} \tag{29}$$

Assembling the global body forces vector of the element yields:

$$\boxed{f^e = -\frac{2\pi}{9}a^2b \begin{bmatrix} 0 \\ -g \\ 0 \\ -g \\ 0 \\ -g \end{bmatrix}} \tag{30}$$

The numerical integration consists on assuming that the body forces are evenly distributed between all the nodes of the element, which is generally a good approximation under normal circumstances.

3 References

1. Hurtado Gómez, J.E. Análisis Matricial de Estructuras. Universidad Nacional de Colombia.

2. Zienkiewicz, O.C., Taylor, R.L. The Finite Element Method, Fifth Edition. Butterworth Heinemann, 2000.