Universitat Politècnica de Catalunya

Escola Tècnica Superior d'Enginyers de Camins, Canals i Ports

MASTER EN INGENIERÍA ESTRUCTURAL Y DE LA CONSTRUCCIÓN

Asignatura:

COMPUTATIONAL STRUCTURAL MECHANICS AND DYNAMICS

Assignment 4

On "Structures of revolution"

By

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Assignment 4.1:

On “Structures of revolution”:

1. Compute the entries of $\boldsymbol{K}^e$ for the following axisymmetric triangle:

$$r_1 = 0, r_2 = r_3 = a, z_1 = z_2 = 0, z_3 = b$$

The material is isotropic with $\nu = 0$ for which the stress-strain matrix is

$$\boldsymbol{E} = E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

2. Show that the sum of the rows (and columns) 2, 4 and 6 of $\boldsymbol{K}^e$ must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.

3. Compute the consistent force vector $\boldsymbol{f}^e$ for gravity forces $\boldsymbol{b} = [0, -g]^T$.

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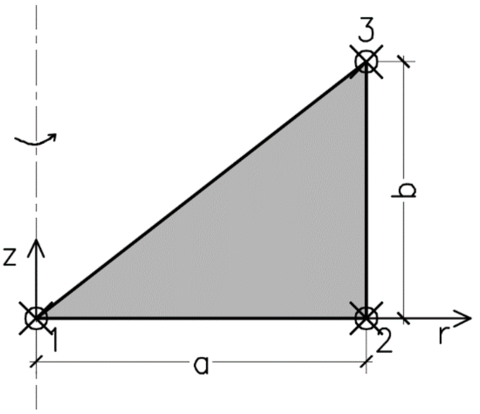
1. The geometry results



The node displacement vector results

$$\boldsymbol{u}_e = \begin{bmatrix} u_{r1} \\ u_{z1} \\ u_{r2} \\ u_{z2} \\ u_{r3} \\ u_{z3} \end{bmatrix}$$

The shape functions are the triangular coordinates:

$$N_i = \xi_i; i = 1,2,3$$

An iso-P element with 3 nodes is defined by

$$\begin{bmatrix} 1 \\ r \\ z \\ u_r \\ u_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \\ u_{r1} & u_{r2} & u_{r3} \\ u_{z1} & u_{z2} & u_{z3} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

The first equations define the geometry of the triangle

$$\begin{bmatrix}1\\r\\z\end{bmatrix}=\begin{bmatrix}1&1&1\\r_1&r_2&r_3\\z_1&z_2&z_3\end{bmatrix}\begin{bmatrix}\xi_1\\\xi_2\\\xi_3\end{bmatrix}$$

Operating

$$1=\xi_1+\xi_2+\xi_3$$

$$r=\xi_1 r_1+\xi_2 r_2+\xi_3 r_3$$

$$z=\xi_1 z_1+\xi_2 z_2+\xi_3 z_3$$

And then

$$\frac{\partial r}{\partial \xi_i}=r_i;\ \frac{\partial z}{\partial \xi_i}=z_i$$

Multiplying by the inverse of the matrix at left on both sides of the equation

$$\begin{bmatrix}1&1&1\\r_1&r_2&r_3\\z_1&z_2&z_3\end{bmatrix}^{-1}\begin{bmatrix}1\\r\\z\end{bmatrix}=\begin{bmatrix}1&1&1\\r_1&r_2&r_3\\z_1&z_2&z_3\end{bmatrix}^{-1}\begin{bmatrix}1&1&1\\r_1&r_2&r_3\\z_1&z_2&z_3\end{bmatrix}\begin{bmatrix}\xi_1\\\xi_2\\\xi_3\end{bmatrix}$$

Doing

$$r_1=0, r_2=r_3=a, z_1=z_2=0, z_3=b$$

$$\begin{bmatrix}\xi_1\\\xi_2\\\xi_3\end{bmatrix}=\begin{bmatrix}1&1&1\\r_1&r_2&r_3\\z_1&z_2&z_3\end{bmatrix}^{-1}\begin{bmatrix}1\\r\\z\end{bmatrix}=\begin{bmatrix}1&1&1\\0&a&a\\0&0&b\end{bmatrix}^{-1}\begin{bmatrix}1\\r\\z\end{bmatrix}=\begin{bmatrix}1&-\frac{1}{a}&0\\0&\frac{1}{a}&-\frac{1}{b}\\0&0&\frac{1}{b}\end{bmatrix}\begin{bmatrix}1\\r\\z\end{bmatrix}$$

Results

$$\xi_1=1-\frac{r}{a};\xi_2=\frac{r}{a}-\frac{z}{b};\ \xi_3=\frac{z}{b}$$

And then

$$\frac{\partial \xi_1}{\partial r}=-\frac{1}{a};\frac{\partial \xi_2}{\partial r}=\frac{1}{a};\frac{\partial \xi_3}{\partial r}=0$$

$$\frac{\partial \xi_1}{\partial z}=0;\frac{\partial \xi_2}{\partial z}=-\frac{1}{b};\frac{\partial \xi_3}{\partial z}=\frac{1}{b}$$

The strain-displacement matrix is

$$
\boldsymbol{B} = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} \end{bmatrix} = \begin{bmatrix} -\frac{1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b} & 0 & \frac{1}{b} \\ \frac{1}{r}-\frac{1}{a} & 0 & \frac{1}{a}-\frac{z}{b\,r} & 0 & \frac{z}{b\,r} & 0 \\ 0 & -\frac{1}{a} & -\frac{1}{b} & \frac{1}{a} & \frac{1}{b} & 0 \end{bmatrix}
$$

The stiffness matrix is

$$
\boldsymbol{K}^e = \int_{\Omega^e} r\boldsymbol{B}^T\boldsymbol{E}\boldsymbol{B}d\Omega = \int_0^a \int_0^{\frac{b}{a}r} r\,\boldsymbol{B}^T\boldsymbol{E}\boldsymbol{B}dzdr
$$

Where

$$
r\boldsymbol{B}^T\boldsymbol{E}\boldsymbol{B} =
$$

$$
\begin{bmatrix}
r\left(\frac{1}{a^2}+\left(\frac{1}{r}-\frac{1}{a}\right)^2\right) & 0 & r\left(-\frac{1}{a^2}+\left(\frac{1}{r}-\frac{1}{a}\right)\left(\frac{1}{a}-\frac{z}{b\,r}\right)\right) & 0 & \frac{\left(\frac{1}{r}-\frac{1}{a}\right)z}{b} & 0 \\
0 & \frac{1}{2}\frac{r}{a^2} & \frac{1}{2}\frac{r}{a\,b} & -\frac{1}{2}\frac{r}{a^2} & -\frac{1}{2}\frac{r}{a\,b} & 0 \\
r\left(-\frac{1}{a^2}+\left(\frac{1}{r}-\frac{1}{a}\right)\left(\frac{1}{a}-\frac{z}{b\,r}\right)\right) & \frac{1}{2}\frac{r}{a\,b} & r\left(\frac{1}{a^2}+\left(\frac{1}{a}-\frac{z}{b\,r}\right)^2+\frac{1}{2\,b^2}\right) & -\frac{1}{2}\frac{r}{a\,b} & r\left(\frac{\left(\frac{1}{a}-\frac{z}{b\,r}\right)z}{b\,r}-\frac{1}{2\,b^2}\right) & 0 \\
0 & -\frac{1}{2}\frac{r}{a^2} & -\frac{1}{2}\frac{r}{a\,b} & r\left(\frac{1}{b^2}+\frac{1}{2\,a^2}\right) & \frac{1}{2}\frac{r}{a\,b} & -\frac{r}{b^2} \\
\frac{\left(\frac{1}{r}-\frac{1}{a}\right)z}{b} & -\frac{1}{2}\frac{r}{a\,b} & r\left(\frac{\left(\frac{1}{a}-\frac{z}{b\,r}\right)z}{b\,r}-\frac{1}{2\,b^2}\right) & \frac{1}{2}\frac{r}{a\,b} & r\left(\frac{z^2}{b^2\,r^2}+\frac{1}{2\,b^2}\right) & 0 \\
0 & 0 & 0 & -\frac{r}{b^2} & 0 & \frac{r}{b^2}
\end{bmatrix}
$$

The previous integral gives the exact solution.
Making a numerically integration, using the centroid point of the triangle, calculating the function there and make that value constant in all the area, we obtain

$$
\boldsymbol{F}(r,z) = r\boldsymbol{B}^T\boldsymbol{E}\boldsymbol{B}
$$

$$
\boldsymbol{K}^e = \int_{\Omega^e} r\boldsymbol{B}^T\boldsymbol{E}\boldsymbol{B}d\Omega = \int_{\Omega^e} \boldsymbol{F}(r,z)d\Omega \approx \Omega^e\,\boldsymbol{F}\left(r=\frac{2a}{3}, z=\frac{b}{2}\right) = \frac{ab}{2}\boldsymbol{F}\left(r=\frac{2a}{3}, z=\frac{b}{2}\right)
$$

$$\begin{bmatrix} \frac{5}{12} b\,Es & 0 & -\frac{7}{24} b\,Es & 0 & \frac{1}{8} b\,Es & 0 \\ 0 & \frac{1}{6} b\,Es & \frac{1}{6} a\,Es & -\frac{1}{6} b\,Es & -\frac{1}{6} a\,Es & 0 \\ -\frac{7}{24} b\,Es & \frac{1}{6} a\,Es & \frac{1}{3} a^2 b\,Es \left(\frac{17}{16\,a^2} + \frac{1}{2\,b^2}\right) & -\frac{1}{6} a\,Es & \frac{1}{3} a^2 b\,Es \left(\frac{3}{16\,a^2} - \frac{1}{2\,b^2}\right) & 0 \\ 0 & -\frac{1}{6} b\,Es & -\frac{1}{6} a\,Es & \frac{1}{3} a^2 b\,Es \left(\frac{1}{b^2} + \frac{1}{2\,a^2}\right) & \frac{1}{6} a\,Es & -\frac{1}{3} \frac{a^2\,Es}{b} \\ \frac{1}{8} b\,Es & -\frac{1}{6} a\,Es & \frac{1}{3} a^2 b\,Es \left(\frac{3}{16\,a^2} - \frac{1}{2\,b^2}\right) & \frac{1}{6} a\,Es & \frac{1}{3} a^2 b\,Es \left(\frac{9}{16\,a^2} + \frac{1}{2\,b^2}\right) & 0 \\ 0 & 0 & 0 & -\frac{1}{3} \frac{a^2\,Es}{b} & 0 & \frac{1}{3} \frac{a^2\,Es}{b} \end{bmatrix}$$

Where Es is E (escalar value)

2. With a rigid body motion in z direction, the displacement vector is

$$\boldsymbol{u}_{RBM,z} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Multiplying with $\boldsymbol{K}^e$ must to be zero, because there are not deformation in the body, so the energy is zero

$$\boldsymbol{u}_{RBM,z}\,\boldsymbol{K}^e = \begin{bmatrix} 0 \\ \frac{1}{6} bE - \frac{1}{6} bE + 0 \\ \frac{1}{6} aE - \frac{1}{6} aE + 0 \\ -\frac{1}{6} bE + \frac{1}{3}\frac{a^2 E}{b} + \frac{1}{6} bE - \frac{1}{3}\frac{a^2 E}{b} \\ -\frac{1}{6} aE + \frac{1}{6} aE \\ -\frac{1}{3}\frac{a^2 E}{b} + \frac{1}{3}\frac{a^2 E}{b} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If consider now

$$\boldsymbol{u}_{RBM,r} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

The result is

$$\left[\frac{1}{4}\,b\,Es, 0, -\frac{7}{24}\,b\,Es + \frac{1}{3}\,a^2\,b\,Es\left(\frac{17}{16\,a^2} + \frac{1}{2\,b^2}\right) + \frac{1}{3}\,a^2\,b\,Es\left(\frac{3}{16\,a^2} - \frac{1}{2\,b^2}\right), 0, \frac{1}{8}\,b\,Es + \frac{1}{3}\,a^2\,b\,Es\left(\frac{3}{16\,a^2} - \frac{1}{2\,b^2}\right) + \frac{1}{3}\,a^2\,b\,Es\left(\frac{9}{16\,a^2} + \frac{1}{2\,b^2}\right), 0\right]$$

Now the sum doesn't vanish. If the sum vanish implies rigid body motion, and that is not true in this case. Because the symmetric revolution condition of the structure, if the 3 nodes of the triangle moves the same value in r direction means the ring is expanding or contracting. That's a deformation of the structure, not a rigid body motion as in the case of the z direction. So there is internal energy involved in the process, implies

$$\boldsymbol{u}_{RBM,z}\,\boldsymbol{K}^e\ must\ be \neq 0$$

3. The consistent force vector is

$$\boldsymbol{f}^e = \int_{\Omega^e} \boldsymbol{N}^T \boldsymbol{b}\, r\, d\Omega$$

$$\boldsymbol{N} = \begin{bmatrix} 1-\frac{r}{a} & \frac{r}{a}-\frac{z}{b} & \frac{z}{b} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\frac{r}{a} & \frac{r}{a}-\frac{z}{b} & \frac{z}{b} \end{bmatrix}$$

So

$$\boldsymbol{N}^T \boldsymbol{b}\, r = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -gr\left(1-\frac{r}{a}\right) \\ -gr\left(\frac{r}{a}-\frac{z}{b}\right) \\ -\frac{grz}{b} \end{bmatrix}$$

Integrating

$$\boldsymbol{f}^e = \int_0^a \int_0^{\frac{b}{a}r} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -gr\left(1-\frac{r}{a}\right) \\ -gr\left(\frac{r}{a}-\frac{z}{b}\right) \\ -\frac{grz}{b} \end{bmatrix} dzdr = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{12}ga^2b \\ -\frac{1}{8}ga^2b \\ -\frac{1}{8}ga^2b \end{bmatrix}$$