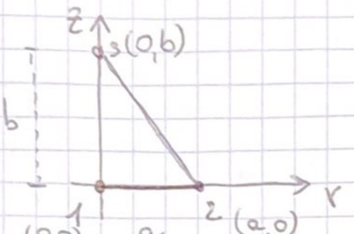
Assignment 4 – Student: Marcello Rubino

1)



Shape functions: (0,0)

$$N_i = \frac{1}{2A^e}\left(m_{1_i} + m_{2_i} r + m_{3_i} z\right) \quad , \quad i = 1 \dots 3$$

$$2A^e = \frac{ab}{2} \cdot 2 = ab \quad \rightarrow \quad m_{1_i} = r_j z_k - z_j r_k \quad m_{2_i} = z_j - z_k \quad m_{3_i} = r_k - r_j$$

So that:

2-3
$$\begin{cases} m_{1_1} = r_2 z_3 - z_2 r_3 = ab \\ m_{2_1} = z_2 - z_3 = -b \\ m_{3_1} = r_3 - r_2 = a - a = \emptyset \end{cases} \rightarrow N_1 = \frac{ab - br}{ab} = \frac{a-r}{a}$$

3-1
$$\begin{cases} m_{1_2} = r_3 z_1 - z_3 r_1 = \emptyset \\ m_{2_2} = z_3 - z_1 = b \\ m_{3_2} = r_1 - r_3 = -a \end{cases} \rightarrow N_2 = \frac{br - az}{ab}$$

1-2
$$\begin{cases} m_{1_3} = r_1 z_2 - z_1 r_2 = \emptyset \\ m_{2_3} = z_1 - z_2 = \emptyset \\ m_{3_3} = r_2 - r_1 = a \end{cases} \rightarrow N_3 = \frac{az}{ab} = \frac{z}{b}$$

So the $\underline{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$ which refers to the nodal displacements

Knowing that $\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{zz} \\ \varepsilon_{\theta\theta} \\ \gamma_{rz} \end{bmatrix} = \underline{D}\,\underline{u} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ 1/r & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} u_r \\ u_z \end{bmatrix}$

and $\underline{u} = \underline{N}\,\underline{u}^e \Rightarrow \underline{\varepsilon} = \underline{D}\,\underline{N}\,\underline{u}^e = \underline{B}\,\underline{u}^e$

where $\underline{B} = \underline{D}\,\underline{N} = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_4}{\partial z} \end{bmatrix}$

in order to compute $\underline{K}^e = \int_{\Omega^e} r\, \underline{B}^T \underline{E}\, \underline{B}\, d\Omega$

$$\underline{B} = \frac{1}{ab}\begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ \frac{ab-br}{r} & 0 & \frac{br-az}{r} & 0 & \frac{az}{r} & 0 \\ 0 & -b & -a & b & a & 0 \end{bmatrix}$$

$$\underline{B}^T \underline{E} = \frac{E}{ab}\begin{bmatrix} -b & 0 & \frac{ab-br}{r} & 0 \\ 0 & 0 & 0 & -b \\ b & 0 & \frac{br-az}{r} & -a \\ 0 & -a & 0 & b \\ 0 & 0 & \frac{az}{r} & a \\ 0 & a & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} = \frac{E}{ab}\begin{bmatrix} -b & 0 & \frac{ab-br}{r} & 0 \\ 0 & 0 & 0 & -b/2 \\ b & 0 & \frac{br-az}{r} & -a/2 \\ 0 & -a & 0 & b/2 \\ 0 & 0 & \frac{az}{r} & a/2 \\ 0 & a & 0 & 0 \end{bmatrix}$$

$$\underline{B}^T \underline{E}\, \underline{B} = \frac{E}{a^2b^2}\begin{bmatrix} b^2+\frac{b^2(a-r)^2}{r^2} & 0 & -b^2+\frac{b(a-r)(br-az)}{r^2} & 0 & \frac{azb(a-r)}{r^2} & 0 \\ & b^2/2 & ab/2 & -b^2/2 & -ab/2 & 0 \\ & & b^2+\frac{(br-az)^2}{r^2}+\frac{a^2}{2} & -\frac{ab}{2} & \frac{az(br-az)}{r^2} & 0 \\ & & & a^2+\frac{b^2}{2} & \frac{ab}{2} & -a^2 \\ & & & & \frac{a^2z^2}{r^2}+\frac{a^2}{2} & 0 \\ & & & & & a^2 \end{bmatrix}$$

$$K_{11} = \int_0^a \int_0^{\frac{br}{a}} r\left[b^2 + \frac{b^2(a-r)^2}{r^2}\right] dz\,dr = \int_0^a \left[r b^2 z + \frac{b^2(a-r)^2 z}{r}\right]_0^{\frac{br}{a}} dr =$$

$$= \int_0^a \frac{b^3}{a} r^2 + \frac{b^3(a-r)^2}{a}\, dr = \int_0^a \frac{b^3 r^2}{a} + \frac{b^3(a^2 - 2ar + r^2)}{a}\, dr =$$

$$= \left[\frac{b^3 r^3}{3a} + b^3 a r - \frac{\cancel{2} b^3 r^2}{\cancel{2}} + \frac{b^3 r^3}{3a}\right]_0^a =$$

$$= \frac{b^3 a^2}{3} + \cancel{b^3 a^2} - \cancel{b^3 a^2} + \frac{b^3 a^2}{3} = \boxed{\frac{2a^2 b^3}{3}}$$

$$K_{13} = K_{31} = \int_0^a \int_0^{\frac{br}{a}} r\left(-b^2 + \frac{b(a-r)(br-az)}{r^2}\right) dz\,dr =$$

$$= \int_0^a \left(\left[-r b^2 z\right]_0^{\frac{br}{a}} + \int_0^{\frac{br}{a}} \frac{(ab-br)(br-az)}{r}\, dz\right) dr =$$

$$= \int_0^a \left(\left[-\frac{r^2 b^3}{a}\right] + \int_0^{\frac{br}{a}} \frac{ab^2 r - b^2 r^2 - a^2 b z + abrz}{r}\, dz\right) dr =$$

$$= \int_0^a -\frac{r^2 b^3}{a} + \left[ab^2 z - b^2 r z - \frac{a^2 b z^2}{2r} + \frac{ab\, z^2}{2}\right]_0^{\frac{br}{a}} dr =$$

$$= \int_0^a \left(-\frac{r^2 b^3}{a} + b^3 r - \frac{b^3 r^2}{a} - \frac{b^3 r}{2} + \frac{b^3 r^2}{2a}\right) dr =$$

$$= \left[-\frac{b^3 r^3}{3a} + \frac{b^3 r^2}{2} - \frac{b^3 r^3}{3a} - \frac{b^3 r^2}{4} + \frac{b^3 r^3}{6a}\right]_0^a = \boxed{-\frac{a^2 b^3}{4}}$$

$$K_{15} = K_{51} = \int_0^a \int_0^{\frac{br}{a}} \cancel{r}\left[\frac{azb(a-r)}{r^{\cancel{2}}}\right] dz\,dr =$$

$$= \int_0^a \left[\frac{a^2 b}{2r} z^2 - \frac{abz^2}{2}\right]_0^{\frac{br}{a}} dr =$$

$$= \int_0^a \left(\frac{\cancel{a^2} b^3 r^{\cancel{2}}}{2\cancel{r}\cancel{a^2}} - \frac{\cancel{a}b^3 r^2}{2a^{\cancel{2}}}\right) dr = \left[\frac{b^3 r^2}{4} - \frac{b^3 r^3}{6a}\right]_0^a = \boxed{\frac{a^2 b^3}{12}}$$

$$K_{22} = -K_{24} = -K_{42} = \int_0^a \int_0^{\frac{br}{a}} r\frac{b^2}{2}\, dz\,dr = \int_0^a \left[\frac{b^2 r z}{2}\right]_0^{\frac{br}{a}} dr =$$

$$= \int_0^a \frac{b^3 r^2}{2a} dr = \left[\frac{b^3 r^3}{6a}\right]_0^a = \boxed{\frac{a^2 b^3}{6}}$$

$$K_{23} = K_{32} = -K_{25} = -K_{52} = K_{45} = K_{54} = \int_0^a \int_0^{\frac{br}{a}} r\frac{ab}{2}\, dz\,dr = \int_0^a \left[\frac{abr}{2} z\right]_0^{\frac{br}{a}} dr =$$

$$= \int_0^a \frac{b^2 r^2}{2} dr = \left[\frac{b^2 r^3}{6}\right]_0^a = \boxed{\frac{a^3 b^2}{6}}$$

$$K_{33} = \int_0^a \int_0^{\frac{br}{a}} r\left[b^2 + \frac{b^2 \cancel{r^2}}{\cancel{r^2}} + \frac{a^2 z^2}{r^2} - \frac{2abrz}{r^2} + \frac{a^2}{2}\right] dz\,dr =$$

$$= \int_0^a \left[b^2 rz + b^2 rz + \frac{a^2 z^3}{3r} - \frac{\cancel{2}abz^2}{\cancel{2}} + \frac{a^2 rz}{2}\right]_0^{\frac{br}{a}} dr =$$

$$= \int_0^a \left(\frac{2b^3 r^2}{a} + \frac{b^3 r^2}{3a} - \frac{b^3 r^2}{a} + \frac{br^2 a}{2}\right) dr =$$

$$= \left[\frac{2b^3 r^3}{3a} + \frac{b^3 r^3}{9a} - \frac{b^3 r^3}{3a} + \frac{br^3 a}{6}\right]_0^a = \boxed{\frac{4a^2 b^3}{9} + \frac{a^4 b}{6}}$$

$$K_{35} = K_{53} = \int_0^a \int_0^{\frac{br}{a}} \cancel{r}\,\frac{az(br-az)}{r^{\cancel{2}}}\, dz\, dr = \int_0^a \left[\frac{ab\cancel{r}z^2}{2\cancel{r}} - \frac{a^2 z^3}{3r} \right]_0^{\frac{br}{a}} dr =$$

$$= \int_0^a \left[\frac{b^3 r^2}{2a} - \frac{b^3 r^2}{3a} \right] dr = \left[\frac{b^3 r^3}{18a} \right]_0^a = \boxed{\frac{a^2 b^3}{18}}$$

$$K_{44} = \int_0^a \int_0^{\frac{br}{a}} \left(a^2 r + \frac{b^2 r}{2} \right) dr\, dz = \int_0^a \left[a^2 r z + \frac{b^2 r z}{2} \right]_0^{\frac{br}{a}} dr =$$

$$= \int_0^a \left(abr^2 + \frac{b^3 r^2}{2a} \right) dr = \left[\frac{abr^3}{3} + \frac{b^3 r^3}{6a} \right]_0^a = \boxed{\frac{a^4 b}{3} + \frac{a^2 b^3}{6}}$$

$$K_{46} = K_{64} = -K_{66} = -\int_0^a \int_0^{\frac{br}{a}} a^2 r\, dr\, dz = -\int_0^a \left[a^2 r z \right]_0^{\frac{br}{a}} dr =$$

$$= -\int_0^a abr^2\, dr = \left[-\frac{abr^3}{3} \right]_0^a = \boxed{-\frac{a^4 b}{3}}$$

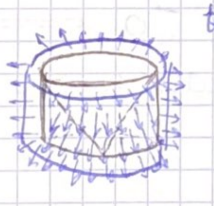
$$K_{55} = \int_0^a \int_0^{\frac{br}{a}} \left[\frac{a^2 z^2}{r} + \frac{a^2 r}{2} \right] dz\, dr = \int_0^a \left[\frac{a^2 z^3}{3r} + \frac{a^2 r z}{2} \right]_0^{\frac{br}{a}} dr =$$

$$= \int_0^a \left(\frac{b^3 r^2}{3a} + \frac{br^2 a}{2} \right) dr = \left[\frac{b^3 r^3}{9a} + \frac{abr^3}{6} \right]_0^a = \boxed{\frac{a^2 b^3}{9} + \frac{a^4 b}{6}}$$

- So the final stiffness matrix:
(simplifying with $\frac{E}{a^2 b^2}$)

$$\underline{K}^e = \begin{bmatrix} \frac{2Eb}{3} & 0 & -\frac{Eb}{4} & 0 & \frac{Eb}{12} & 0 \\ 0 & \frac{Eb}{6} & \frac{Ea}{6} & -\frac{Eb}{6} & -\frac{Ea}{6} & 0 \\ -\frac{Eb}{4} & \frac{Ea}{6} & \frac{E(3a^2+8b^2)}{18b} & -\frac{Ea}{6} & -\frac{E(3a^2-b^2)}{18b} & 0 \\ 0 & -\frac{Eb}{6} & -\frac{Ea}{6} & \frac{E(2a^2+b^2)}{6b} & \frac{Ea}{6} & -\frac{Ea^2}{3b} \\ \frac{Eb}{12} & -\frac{Ea}{6} & -\frac{E(3a^2-b^2)}{18b} & \frac{Ea}{6} & \frac{E(3a^2+2b^2)}{18b} & 0 \\ 0 & 0 & 0 & -\frac{Ea^2}{3b} & 0 & \frac{Ea^2}{3b} \end{bmatrix}$$

2) The stiffness matrix shows particularities: the sum of rows (or columns) 2-4-6 are equal to zero, while the sum of 1-3-5 rows does not vanish; the reason is because the odd rows refer to the radial coordinate dof: an uniform displacement field along "r" does not create a rigid motion along the radial direction; it actually creates non zero internal forces associated to a radial deformation. Differently the "z" direction behaves normally and an uniform field is associated to a rigid motion.



the solid enlarges!

3) gravity force: $\underline{b} = [0, -g]^T$ is a body force: therefore:

$$\underline{f}^e = \int_{\Omega^e} r \underline{N}^T \underline{b} \, d\Omega$$

$$\underline{N}^T = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} = \begin{bmatrix} \frac{a-r}{a} & 0 \\ 0 & \frac{a-r}{a} \\ \frac{1}{ab}(br-az) & 0 \\ 0 & \frac{1}{ab}(br-az) \\ \frac{z}{b} & 0 \\ 0 & \frac{z}{b} \end{bmatrix}$$

$$\underline{f}^e = \int_0^a \int_0^{\frac{br}{a}} \begin{bmatrix} 0 \\ -\frac{rg}{a}(a-r) \\ 0 \\ -\frac{rg}{ab}(br-az) \\ 0 \\ -\frac{zrg}{b} \end{bmatrix} dz\,dr = \int_0^a \begin{bmatrix} 0 \\ -\frac{\cancel{a}}{\cancel{a}} rgz + \frac{r^2 g z}{a} \\ 0 \\ -\frac{r^2 g \cancel{b} z}{a\cancel{b}} + \frac{\cancel{a} r g z^2}{2\cancel{a} b} \\ 0 \\ -\frac{rgz^2}{2b} \end{bmatrix}_0^{\frac{br}{a}} dr =$$

$$= \int_0^a \begin{bmatrix} 0 \\ -\frac{br^2}{a} g + \frac{br^3}{a^2} g \\ 0 \\ -\frac{br^3}{a^2} g + \frac{b^{\cancel{2}} r^3}{2a^2 \cancel{b}} g \\ 0 \\ -\frac{b^{\cancel{2}} r^3}{2\cancel{b} a^2} g \end{bmatrix} dr = \begin{bmatrix} 0 \\ -\frac{br^3}{3a} g + \frac{br^4}{4a^2} g \\ 0 \\ -\frac{br^4}{4a^2} g + \frac{br^4}{8a^2} g \\ 0 \\ -\frac{br^4}{8a^2} g \end{bmatrix}_0^a = \begin{Bmatrix} 0 \\ -\frac{a^2 b}{12} g \\ 0 \\ -\frac{a^2 b}{8} g \\ 0 \\ -\frac{a^2 b}{8} g \end{Bmatrix} = \underline{f}^e$$

as expected there are non-zero components only along the z direction (2-4-6 position)