Universitat Politècnica de Catalunya

Master of Science in Computational Mechanics

Computational Structural Mechanics and Dynamics

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# Assignment 4
# Structures of revolution

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*Authors:*
Carlos Eduardo Ribeiro Santa Cruz Mendoza

March 4, 2019

# 1 Axis-symmetric Triangle

A structure is stated to be axis-symmetric and can be obtained by the revolution of the triangle shown on Figure 1.1. The coordinates system used is the cylindrical, being $\theta$ the axis normal to the triangle.

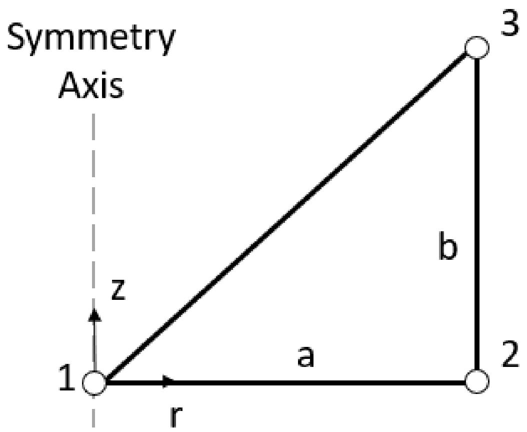


Figure 1.1: Problem geometry

## 1.1 Stiffness Matrix

In order to find the relation between forces and displacements $\boldsymbol{Ku} = f$ or, in other words, to find the stiffness matrix $\boldsymbol{K}$, we must select the relevant strain-displacement relations. Since the problem is axis-symmetric, there can be no variable variation on the $\theta$ direction. Thus, the strain tensor can be written as in Equation 1.1.

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{rz} & 0 \\ \varepsilon_{rz} & \varepsilon_{zz} & 0 \\ 0 & 0 & \varepsilon_{\theta\theta} \end{bmatrix} \Rightarrow \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{zz} \\ \varepsilon_{\theta\theta} \\ 2\varepsilon_{rz} \end{bmatrix} = \begin{bmatrix} \partial/\partial r & 0 \\ 0 & \partial/\partial z \\ 1/r & 0 \\ \partial/\partial z & \partial/\partial r \end{bmatrix} \begin{bmatrix} u_r \\ u_z \end{bmatrix} = \boldsymbol{Du} \tag{1.1}$$

From the strain definition, the stress can be obtained through the constitutive matrix by $\boldsymbol{\sigma} = \boldsymbol{E\varepsilon}$. Where the constitutive matrix for the given problem (isotropic and with $\nu = 0$) is:

$$\boldsymbol{E} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \tag{1.2}$$

Using the isoparametric formulation to define the geometry and interpolate the displacements we obtain (since we're dealing with a one-element domain, the upper script $(e)$ was suppressed):

$$\begin{bmatrix} 1 \\ r \\ z \\ u_r \\ u_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \\ u_{r1} & u_{r2} & u_{r3} \\ u_{z1} & u_{z2} & u_{z3} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & a & a \\ 0 & 0 & b \\ u_{r1} & u_{r2} & u_{r3} \\ u_{z1} & u_{z2} & u_{z3} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ 1-\xi-\eta \end{bmatrix} \tag{1.3}$$

where $\xi$ and $\eta$ are the natural coordinates.

The strain is now defined in terms of the displacements and the shape function. Consequently we must evaluate the shape functions and their derivatives on every nodal point in terms of $r$ and $z$. However, the shape functions are given on natural coordinates, needing, thus, a coordinate transformation.

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & a & a \\ 0 & 0 & b \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ r \\ z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \partial N_i/\partial r \\ \partial N_i/\partial z \end{bmatrix} = \boldsymbol{J}^{-1} \begin{bmatrix} \partial N_i/\partial \xi \\ \partial N_i/\partial \eta \end{bmatrix} \tag{1.4}$$

The matrix $\boldsymbol{J}$ is the Jacobian, that for the given problem is defined by:

$$\boldsymbol{J} = \begin{bmatrix} \partial r/\partial \xi & \partial z/\partial \xi \\ \partial r/\partial \eta & \partial z/\partial \eta \end{bmatrix} = \begin{bmatrix} -a & -b \\ 0 & -b \end{bmatrix} \tag{1.5}$$

The derivatives of the shape functions in respect to the natural coordinate can be calculated from their definitions stated in Equation 1.3:

$$\begin{bmatrix} \partial N_1/\partial \xi \\ \partial N_1/\partial \eta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} \partial N_2/\partial \xi \\ \partial N_2/\partial \eta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} \partial N_3/\partial \xi \\ \partial N_3/\partial \eta \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \tag{1.6}$$

Additionally, we can define the vector N as:

$$\boldsymbol{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \tag{1.7}$$

The definitions stated in equations 1.3 to 1.7 allow us to obtain the shape functions in terms of global coordinates $r$ and $z$, as well as to calculate the $\boldsymbol{B} = \boldsymbol{DN}$ matrix, yielding:

$$N_1 = 1 - \frac{r}{a} \quad N_2 = \frac{r}{a} - \frac{z}{b} \quad N_3 = \frac{z}{b} \tag{1.8}$$

$$\boldsymbol{B} = \begin{bmatrix} -1/a & 0 & 1/a & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/b & 0 & 1/b \\ {}^1\!/_r - {}^1\!/_a & 0 & {}^1\!/_a - {}^z\!/_{rb} & 0 & {}^z\!/_{rb} & 0 \\ 0 & -{}^1\!/_a & -{}^1\!/_b & {}^1\!/_a & {}^1\!/_b & 0 \end{bmatrix} \tag{1.9}$$

Finally, the stiffness matrix can be obtained by the relation in Equation 1.10. The axis-symmetry of the problem allows the volume integral of a 3D element to be reduced to an area integral (on $r$ and $z$ directions) multiplied by the circumference.

$$\boldsymbol{K} = 2\pi \int_{\Omega} r\boldsymbol{B}^T\boldsymbol{E}\boldsymbol{B}dA = 2\pi\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} r\boldsymbol{B}^T\boldsymbol{E}\boldsymbol{B}det\boldsymbol{J}d\xi d\eta \tag{1.10}$$

To integrate the stiffness matrix we resort to the one-point Gauss integration, in which the evaluated function must be integrated on the middle (zero) of the natural domain, being the centroid of the triangular element for the given problem ($r = {}^{2a}\!/_3$ and $z = {}^b\!/_3$). Numerically integrating the stiffness matrix at the centroid also avoids the numerical issues with the ${}^1\!/_r$ terms present on the $\boldsymbol{B}$ matrix when $r = 0$ at point 1.

The final relation is then given by (suppressing the $2\pi$ term, for it cancels out with the same term on the r.h.s of $\boldsymbol{Ku} = \boldsymbol{f}$):

$$\boldsymbol{K} = \frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{p} w_k w_l r_{k,l}\boldsymbol{B}_{k,l}^T\boldsymbol{E}\boldsymbol{B}_{k,l}det\boldsymbol{J}_{k,l} \tag{1.11}$$

where $p = 1$ and the weights $w_k$ and $w_l$ for the one-point Gauss integration are both equal to 1. The stiffness matrix for the axis-symmetric triangular element is, thus:

$$\boldsymbol{K} = E\begin{bmatrix} {}^{5b}\!/_{12} & 0 & -{}^b\!/_4 & 0 & {}^b\!/_{12} & 0 \\ 0 & {}^b\!/_6 & {}^a\!/_6 & -{}^b\!/_6 & -{}^a\!/_6 & 0 \\ -{}^b\!/_4 & {}^a\!/_6 & {}^{5b}\!/_{12} + {}^{a^2}\!/_{6b} & -{}^a\!/_6 & {}^b\!/_{12} - {}^{a^2}\!/_{6b} & 0 \\ 0 & -{}^b\!/_6 & -{}^a\!/_6 & {}^b\!/_6 + {}^{a^2}\!/_{3b} & {}^a\!/_6 & -{}^{a^2}\!/_{3b} \\ {}^b\!/_{12} & -{}^a\!/_6 & {}^b\!/_{12} - {}^{a^2}\!/_{6b} & {}^a\!/_6 & {}^b\!/_{12} + {}^{a^2}\!/_{6b} & 0 \\ 0 & 0 & 0 & -{}^{a^2}\!/_{3b} & 0 & {}^{a^2}\!/_{3b} \end{bmatrix} \tag{1.12}$$

## 1.2 Rigid body motion

The sum of the rows 2, 4 and 6 of the stiffness matrix (Equation 1.15) are equal to zero. This is due to the possibility, within this model, for the element to have a rigid body motion in the $z$-direction.

The equations of the linear system $\boldsymbol{Ku} = \boldsymbol{f}$ formed by rows 2, 4 and 6 are related to the displacements in the $z$-direction of every node. If the element is under a rigid body motion with no external forces on the $z$-direction, the displacement values in rows 2, 4 and 6 will all have the same value and $\boldsymbol{f} = \mathbf{0}$, thus for the linear system to hold the coefficients in the stiffness matrix must cancel out, as they do on the given case.

On the other hand, a rigid body motion in the $r$-direction is not possible within this model due to the symmetry hypothesis. The symmetry would be lost in respect to the chosen axis if there were any rigid body motion, consequently the linear system doesn't support such inputs. This is also verified by the fact that the sum of rows 1, 3 and 5 of the stiffness matrix don't cancel out.

## 1.3 Force vector

Given that the body is under gravitational forces given by

$$\boldsymbol{b} = \begin{bmatrix} 0 \\ -g \end{bmatrix} \tag{1.13}$$

Similarly to the stiffness matrix, the consistent force vector can also be calculated via Gauss integration, as presented in Equation 1.14.

$$\boldsymbol{f} = \frac{1}{2} \sum_{k=1}^{p} \sum_{l=1}^{p} w_k w_l \boldsymbol{N}_{k,l}^T \boldsymbol{b} r_{k,l} det \boldsymbol{J}_{k,l} \tag{1.14}$$

Yielding for the given problem:

$$\boldsymbol{f} = \begin{bmatrix} 0 \\ -\,{}^{a^2bg}\!/_{9} \\ 0 \\ -\,{}^{a^2bg}\!/_{9} \\ 0 \\ -\,{}^{a^2bg}\!/_{9} \end{bmatrix} \tag{1.15}$$

# 2 Isoparametric representation

In order to find the appropriate shape functions for the quadrilateral element shown on Figure 2.1, the line-product method is initially applied for the $N_5$ shape function, also depicted on the Figure 2.1.

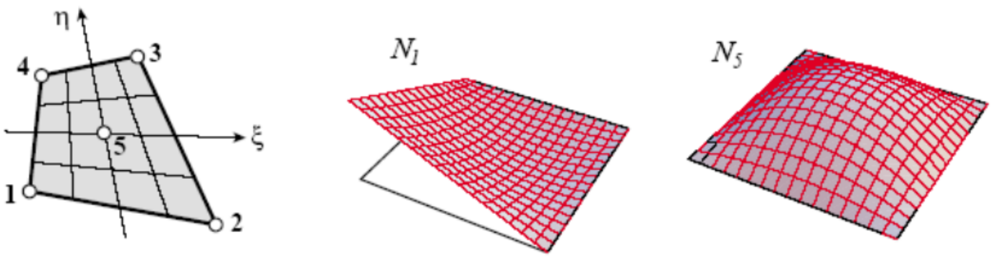


Figure 2.1: Five-node quadrilateral element

Following the aforementioned method, $N_5(\xi, \eta)$ has the form:

$$N_5(\xi, \eta) = c_5 L_{1-2} L_{2-3} L_{3-4} L_{4-1} \tag{2.1}$$

The equation $L_{1-2}$ for the side 1-2 is $\eta + 1 = 0$, for the side 2-3 is $\xi - 1 = 0$, for the side 3-4 is $\eta - 1 = 0$ and for the side 4-1 is $\xi + 1 = 0$. Substituting on Equation 2.1 yields:

$$N_5(\xi, \eta) = c_5(\eta + 1)(\xi - 1)(\eta - 1)(\xi - 1) \tag{2.2}$$

Equation 2.2 takes the value zero over all nodes from 1 to 4. For the shape function to have a value of one at the node 5, the equation should be normalized:

$$N_5(0, 0) = c_5(0 + 1)(0 - 1)(0 - 1)(0 - 1) \tag{2.3}$$

Thus, $c_5$ must take the value of 1, yielding a final expression for $N_5$:

$$N_5(\xi, \eta) = (\eta - 1)^2(\xi - 1)^2 \tag{2.4}$$

Following the hint given on the problem description, we know that the remaining shape functions will take the form shown on Equation 2.5.

$$N_i = N_i^* + \alpha N_5 \qquad \text{for} \quad i = 1, 2, 3, 4 \tag{2.5}$$

where $N_i^*$ are the shape functions of the 4-noded quadrilateral and $\alpha$ is responsible for making $N_i = 0$ on node 5.

Applying formula 2.5 for every $i$ yields:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) + \alpha(1-\xi^2)(1-\eta^2) \tag{2.6}$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta) + \alpha(1-\xi^2)(1-\eta^2) \tag{2.7}$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta) + \alpha(1-\xi^2)(1-\eta^2) \tag{2.8}$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta) + \alpha(1-\xi^2)(1-\eta^2) \tag{2.9}$$

But we know that on node 5 $N_1 = N_2 = N_3 = N_4 = 0$ and that $\xi = \eta = 0$, yielding an alpha (for all cases) of:

$$0 = \frac{1}{4} + \alpha \quad \Rightarrow \quad \alpha = -\frac{1}{4} \tag{2.10}$$

Thus, the remaining shape functions take the form:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2) = -\frac{1}{4}(1-\xi)(1-\eta)(\xi+\eta+\xi\eta) \tag{2.11}$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2) = \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta+\xi\eta) \tag{2.12}$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2) = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-\xi\eta) \tag{2.13}$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2) = \frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta+\xi\eta) \tag{2.14}$$

## 2.1 Compatibility

Compatibility can be verified by evaluating the degree of the polynomial used to interpolate between nodes. A side with two nodes, such as 1-2 for example, must yield a linear interpolation. Evaluating the shape function $N_1$ At side 1-2, where $\eta = -1$ we obtain:

$$N_1 = -\frac{1}{4}(1-\xi)(1+1)(\xi-1-\xi) = \frac{1}{2}(1-\xi) \tag{2.15}$$

As expected, the interpolating function of side 1-2 is linear. Similarly, linear interpolations can be found for all sides of the quadrilateral, verifying compatibility.

## 2.2 Unity sum

The unity sum of shape functions can be verified by remembering the general form given on Equation 2.5. The sum of all shape functions yields:

$$\sum_{i=1}^{5} N_i = \sum_{i=1}^{4} N_i^* + 4\alpha N_5 + N_5 = \sum_{i=1}^{4} N_i^* + 4(-\frac{1}{4})N_5 + N_5 = \sum_{i=1}^{4} N_i^* \tag{2.16}$$

But the $N_i^*$ shape functions are from the 4-noded quadrilateral element, whose sum has already been proven to be equal to 1. Consequently, from the result on the r.h.s of Equation 2.16, the unity sum for the 5-noded quadrilateral element is automatically verified.