Master of Science in Computational Mechanics

Computational Structural Mechanics and Dynamics

Assignment 4: Structures of revolution and Isoparametric representation

Submitted By:
Mario Alberto Méndez Soto

Submitted To:
Prof. Miguel Cervera

Spring Semester, 2019

Assignment 4.1 - Structures of revolution

1. **Compute the entries of $\mathbf{K^e}$ for the following axisymmetric triangle:**

$$r_1 = 0, \qquad r_2 = r_3 = a, \qquad z_1 = z_2 = 0, \qquad z_3 = b$$

The material is isotropic with $\nu = 0$ for which the stress-strain matrix is,

$$\mathbf{E} = E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Figure (1) shows the geometric representation of an element of the cross section of the given axisymmetric problem.

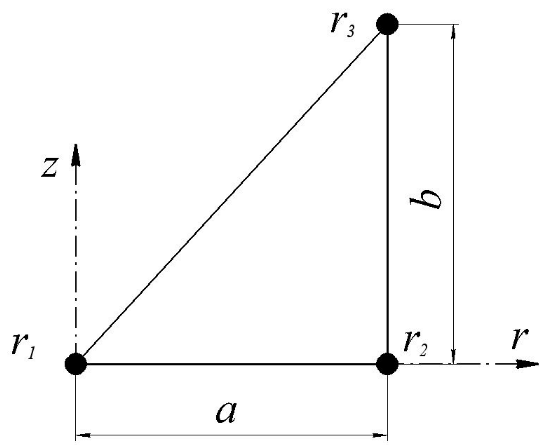

Figure 1 – Discretization of one axisymmetric triangle

The stiffness matrix of an axisymmetric linear triangular element is defined as:

$$\mathbf{K^e} = 2\pi \int_A r\mathbf{B}^T\mathbf{E}\mathbf{B}dA \tag{1}$$

where the matrix $\mathbf{B}$ can be expressed as follows:

$$\mathbf{B} = \frac{1}{2A} \left[\begin{array}{cc|cc|cc} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} \end{array} \right] \tag{2}$$

For an axisymmetric triangle, its shape functions are:

$$N_i = \frac{1}{2A}(a_i + b_i r + c_i z) \tag{3}$$

where:

$$
\begin{aligned}
a_i &= r_j z_k - r_k z_j \\
b_i &= z_j - z_k \\
c_i &= r_k - r_j
\end{aligned}
$$

Considering Figure (1), the previously introduced constants are computed.

Node	r	z	a_i	b_i	c_i
1	0	0	ab	-b	0
2	a	0	0	b	-a
3	a	b	0	0	a

As a result and considering that $A = \frac{1}{2}ab$, the shape functions become:

$$
\begin{aligned}
N_1 &= \frac{1}{2A}(ab - br) = 1 - r/a \\
N_2 &= \frac{1}{2A}(br - az) = r/a - z/b \\
N_3 &= \frac{1}{2A}(az) = z/b
\end{aligned}
$$

Computing the corresponding derivatives and substituting into equation (2), matrix $\mathbf{B}$ reads:

$$
\mathbf{B} = \frac{1}{2A}\left[\begin{array}{cc|cc|cc}
-b & 0 & b & 0 & 0 & 0 \\
0 & 0 & 0 & -a & 0 & a \\
\frac{ab}{r} - b & 0 & b - \frac{az}{r} & 0 & \frac{az}{r} & 0 \\
0 & -b & -a & b & a & 0
\end{array}\right] = \left[\begin{array}{cc|cc|cc}
-\frac{1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{b} & 0 & \frac{1}{b} \\
\frac{1}{r} - \frac{1}{a} & 0 & \frac{1}{a} - \frac{z}{rb} & 0 & \frac{z}{rb} & 0 \\
0 & -\frac{1}{a} & -\frac{1}{b} & \frac{1}{a} & \frac{1}{b} & 0
\end{array}\right] \tag{4}
$$

Substituting the obtained expressions into equation (1), the expression for $\mathbf{K^e}$ yields:

$$
\int_A 2\pi E r \begin{bmatrix}
-\frac{1}{a} & 0 & \frac{1}{r} - \frac{1}{a} & 0 \\
0 & 0 & 0 & -\frac{1}{a} \\
\frac{1}{a} & 0 & \frac{1}{a} - \frac{z}{rb} & -\frac{1}{b} \\
0 & -\frac{1}{b} & 0 & \frac{1}{a} \\
0 & 0 & \frac{z}{rb} & \frac{1}{b} \\
0 & \frac{1}{b} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1/2
\end{bmatrix}
\left[\begin{array}{cc|cc|cc}
-\frac{1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{b} & 0 & \frac{1}{b} \\
\frac{1}{r} - \frac{1}{a} & 0 & \frac{1}{a} - \frac{z}{rb} & 0 & \frac{z}{rb} & 0 \\
0 & -\frac{1}{a} & -\frac{1}{b} & \frac{1}{a} & \frac{1}{b} & 0
\end{array}\right] dA
$$

Additionally, if an isoparametric formulation is used, the strain and geometric coordinates can be approximated with the following expression:

$$\begin{bmatrix} 1 \\ r \\ z \\ u_r \\ u_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \\ u_{r_1} & u_{r_2} & u_{r_3} \\ u_{z_1} & u_{z_2} & u_{z_3} \end{bmatrix} \begin{bmatrix} N_1 = \zeta_1 \\ N_2 = \zeta_2 \\ N_3 = \zeta_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & a & a \\ 0 & 0 & b \\ u_{r_1} & u_{r_2} & u_{r_3} \\ u_{z_1} & u_{z_2} & u_{z_3} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ 1-\xi-\eta \end{bmatrix} \tag{5}$$

where ζ_i, ξ and η are triangular and natural coordinates, respectively.

As a result, equation (1) can be expressed as:

$$\mathbf{K^e} = 2\pi \int_A r\mathbf{B}^T\mathbf{E}\mathbf{B}dA = 2\pi \int_0^a \int_0^b r\mathbf{B}^T\mathbf{E}\mathbf{B}dzdr = 2\pi \int_0^1 \int_0^1 r\mathbf{B}^T\mathbf{E}\mathbf{B} \mid \mathbf{J} \mid d\xi d\eta \tag{6}$$

where the matrix $\mathbf{J}$ is the Jacobian, that for the given problem is defined by:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -a & -b \\ 0 & -b \end{bmatrix}$$

The integration of the stiffness matrix will be performed using an one-point Gauss numerical integration, in which the function must be evaluated at the the centroid of the triangle element ($r = 2a/3$ and $z = b/3$), i.e. $\xi_1 = \xi_2 = \xi_3 = 1/3$. Then, the final expression to compute the stiffness matrix becomes:

$$\mathbf{K}^1 = \frac{1}{2} \sum_{k=1}^{p} \sum_{l=1}^{q} w_k w_l r_{k,l} \mathbf{B}_{k,l}^T \mathbf{B}_{k,l} \mid \mathbf{J}_{k,l} \mid$$

where $p = q = 1$ and the weights w_k and w_l for the one-point Gauss integration are both equal to 1. Thus, the stiffness matrix is equal to:

$$K = E \begin{bmatrix} \frac{5b}{12} & 0 & -\frac{b}{4} & 0 & \frac{b}{12} & 0 \\ 0 & \frac{b}{6} & \frac{a}{6} & -\frac{b}{6} & -\frac{a}{6} & 0 \\ -\frac{b}{4} & \frac{a}{6} & \frac{2a^2+5b^2}{12b} & -\frac{a}{6} & -\frac{2a^2-b^2}{12b} & 0 \\ 0 & -\frac{b}{6} & -\frac{a}{6} & \frac{2a^2+b^2}{6b} & \frac{a}{6} & -\frac{a^2}{3b} \\ \frac{b}{12} & -\frac{a}{6} & -\frac{2a^2-b^2}{12b} & \frac{a}{6} & \frac{2a^2+b^2}{12b} & 0 \\ 0 & 0 & 0 & -\frac{a^2}{3b} & 0 & \frac{a^2}{3b} \end{bmatrix}$$

[1]The term 2π factor is suppressed, since it cancels out with the same term on the r.h.s of $\mathbf{Ku} = \mathbf{f}$. Additionally, the expression is multiplied by 1/2 so that the element area is correctly computed in cases where the weights are not normalized.

2. **Show that the sum of the rows (and columns) 2, 4 and 6 of $\mathbf{K^e}$ must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.**

   For row and column 2:

$$0 + \frac{b}{6} + \frac{a}{6} - \frac{b}{6} - \frac{a}{6} + 0 = 0$$

   For row and column 4:

$$0 - \frac{b}{6} - \frac{a}{6} + \frac{2a^2 + b^2}{6b} + \frac{a}{6} - \frac{a^2}{3b} = 0$$

   For row and column 6:

$$0 + 0 + 0 + -\frac{a^2}{3b} + 0 + \frac{a^2}{3b} = 0$$

   The summation of the rows and columns 2, 4 and 6 is equal to 0 due to the fact that the model allows for a rigid-body motion in the z-direction. If the system of linear equations $\mathbf{Ku} = \mathbf{f}$ is analyzed, it can be seen that the equations 2, 4 and 6 are related to the displacements and forces in the z-direction of each one of the nodes. In the particular case of rigid motion and no external forces in the z-direction, the coefficients in the stiffness matrix must cancel out since the displacement values in rows 2, 4 and 6 will be the same.

   On the other hand, due to the symmetry condition imposed, rigid motion in the r-direction is not possible. This is the reason why, for rows and columns 1, 3 and 5 the coefficients do not vanish.

3. **Compute the consistent force vector f^e for gravity forces $\mathbf{b} = [0, -\mathbf{g}]^T$.**

   The force vector due to the weight of the element can be calculated using the following expression:

$$\mathbf{f}_b = 2\pi \int_A r\mathbf{N^T}\mathbf{b}dA \tag{7}$$

   Considering the body force vector $\mathbf{b} = [0, -\mathbf{g}]^T$ and the previously computed shape functions, the expression (7) reduces to:

$$\mathbf{f}_b = 2\pi \int_A \frac{1}{2A} \begin{bmatrix} ab - br & 0 \\ 0 & ab - br \\ br - az & 0 \\ 0 & br - az \\ az & 0 \\ 0 & az \end{bmatrix} \begin{bmatrix} 0 \\ -g \end{bmatrix} rdA$$

   As done for the stiffness matrix, the integral can be computed using a Gaussian numerical integration:

$$\mathbf{f}_b{}^2 = \frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{q} w_k w_l r_{k,l}\mathbf{N}_{k,l}^T\mathbf{b} \mid \mathbf{J}_{k,l} \mid$$

Thus, the force vector becomes:

$$\mathbf{f}_b = \begin{bmatrix} 0 \\ -\frac{a^2bg}{9} \\ 0 \\ -\frac{a^2bg}{9} \\ 0 \\ -\frac{a^2bg}{9} \end{bmatrix}$$

[2] Once again, the term 2π factor is suppressed, since it cancels out with the same term on the l.h.s of $\mathbf{Ku} = \mathbf{f}$ and the expression is multiplied by $1/2$ so that the element area is correctly computed in cases where the weights are not normalized. The weights are the same used for the stiffness matrix.

Assignment 4.2 - Isoparametric representation

A five node quadrilateral element has the nodal configuration shown if the figure. Perspective views of N_1^e and N_5^e are shown in the same figure.

Find five shape functions N_i^e, $i = 1, \ldots, 5$ that satisfy compatibility and also verify that their sum is unity.

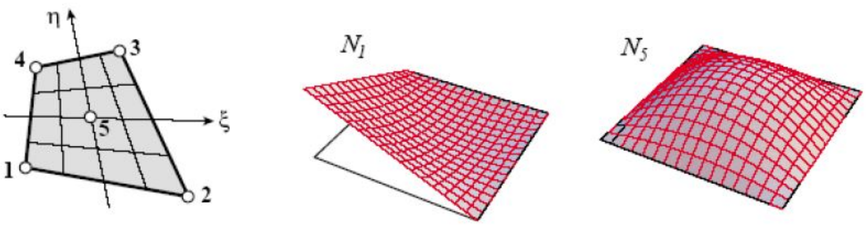

Figure 2 – Five node quadrilateral element

Hint: develop $N_5(\xi, \eta)$ first for the 5-node quad using the line-product method. Then the corner shape functions $\underline{N_i}(\xi, \eta)$, $i = 1, 2, 3, 4$, for the 4-node quad (already given in the notes). Finally combine $N_i = \underline{N_i} + \alpha N_5$ determining α so that all N_i vanish node 5. Check that $N_1 + N_2 + N_3 + N_4 + N_5 = 1$ identically.

Using the line-product method, $N_5(\xi, \eta)$ has the following form:

$$N_5(\xi, \eta) = c_5 L_{1-2} L_{2-3} L_{3-4} L_{4-1}$$

Taking into consideration that for sides $1-2$, $2-3$, $3-4$, and $4-1$, the coordinates are $\xi = -1$ or $\xi = 1$, $\eta = -1$ or $\eta = 1$, $\xi = 1$ or $\xi = -1$ and $\eta = 1$ or $\eta = -1$, respectively. Replacing in the previous expression:

$$N_5(\xi, \eta) = c_5(1 + \eta)(1 - \xi)(1 - \eta)(1 + \xi)$$

which plainly vanishes over nodes 1, 2, 3 and 4. The expression can be normalized by finding the value c_5 that satisfies the expression, it can be obtained that:

$$\begin{gathered} N_5|_{\text{at node 5}} = 1 \\ = c_5(1 + 0)(1 - 0)(1 - 0)(1 + 0) = 1 \\ \Rightarrow c_5 = 1 \end{gathered}$$

Thus,

$$\boxed{N_5(\xi, \eta) = (1 + \eta)(1 - \xi)(1 - \eta)(1 + \xi) = (1 - \xi^2)(1 - \eta^2)}$$

As advised in the hint of the problem, the remaining shape functions can be expressed as:

$$N_i = \underline{N_i} + \alpha N_5$$

where $\underline{N_i}(\xi, \eta)$ is the corresponding shape functions for the 4-noded quadrilateral element and α is determined such that all N_i vanish at node 5.

For node 1:

$$\begin{aligned} N_1 &= \underline{N_1} + \alpha N_5 \\ &= \frac{1}{4}(1-\xi)(1-\eta) + \alpha(1-\xi^2)(1-\eta^2) \end{aligned}$$

At node 5 $\eta = \xi = 0$, whence:

$$\begin{aligned} \frac{1}{4}(1-0)(1-0) + \alpha(1-0^2)(1-0^2) = 0 \\ \Rightarrow \boxed{\alpha = -\frac{1}{4}} \end{aligned}$$

which will be the same for all N_i. Then,

$$\begin{aligned} N_1 &= \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2) \\ &= \frac{1}{4}(1-\xi)(1-\eta)[1-(1+\xi)(1+\eta)] \\ &= \frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-\xi\eta) \\ &= \boxed{-\frac{1}{4}(1-\xi)(1-\eta)(\xi+\eta+\xi\eta)} \end{aligned}$$

For node 2:

$$\begin{aligned} N_2 &= \underline{N_2} - \frac{1}{4}N_5 \\ &= \frac{1}{4}(1+\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2) \\ &= \frac{1}{4}(1+\xi)(1-\eta)[1-(1-\xi)(1+\eta)] \\ &= \boxed{\frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta+\xi\eta)} \end{aligned}$$

For node 3:

$$\begin{aligned} N_3 &= \underline{N_3} - \frac{1}{4}N_5 \\ &= \frac{1}{4}(1+\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2) \\ &= \frac{1}{4}(1+\xi)(1+\eta)[1-(1-\xi)(1-\eta)] \\ &= \boxed{\frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-\xi\eta)} \end{aligned}$$

For node 4:

$$
\begin{aligned}
N_4 &= \underline{N_4} - \frac{1}{4}N_5 \\
&= \frac{1}{4}(1-\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2) \\
&= \frac{1}{4}(1-\xi)(1+\eta)[1-(1+\xi)(1-\eta)] \\
&= \boxed{\frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta+\xi\eta)}
\end{aligned}
$$

In summary, the shape functions for the 5-noded quadrilateral elements are:

$$
\boxed{
\begin{gathered}
N_1 = \frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-\xi\eta) \\
N_2 = \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta+\xi\eta) \\
N_3 = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-\xi\eta) \\
N_4 = \frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta+\xi\eta) \\
N_5 = (1-\xi^2)(1-\eta^2)
\end{gathered}
}
$$

The compatibility check can be performed by analyzing one of the interelement boundary nodes. For instance, node 1 belongs to the boundaries $1-2$ and $1-4$. Over the side $1-2$, $\eta=-1$ and N_1 becomes:

$$
\begin{aligned}
N_1 &= \frac{1}{4}(1-\xi)(1-(-1))(-\xi-(-1)-\xi(-1)) \\
&= \frac{1}{2}(1-\xi)
\end{aligned}
$$

thus N_1 is a linear function of ξ. Similarly, over side $1-4$, $\xi=-1$ and N_1 becomes:

$$
\begin{aligned}
N_1 &= \frac{1}{4}(1-(-1))(1-\eta)(-(-1)-\eta-(-1)\eta) \\
&= \frac{1}{2}(1-\eta)
\end{aligned}
$$

thus N_1 is a linear function of η. Consequently the polynomial variation order is 1 over both sides. Because there are two nodes on each side the compatibility condition is satisfied.

Analogous analyses can be performed for nodes 2,3, and 4.

To verify that the sum of the shape functions is equal to 1, it is possible to consider the general form of the N_i for $i = 1, 2, 3, 4$. Thus,

$$
\begin{aligned}
\sum_{i=1}^{5} N_i &= N_1 + N_2 + N_3 + N_4 + N_5 \\
&= \Big(\sum_{j=1}^{4} \underline{N_j} + \alpha_j N_5 \Big) + N_5
\end{aligned}
$$

since $\alpha_j = \alpha = 1/4$:

$$
\begin{aligned}
&= \sum_{j=1}^{4} \underline{N_j} + \sum_{j=1}^{4} \alpha N_5 + N_5 \\
&= \sum_{j=1}^{4} \underline{N_j} + 4\Big(-\frac{1}{4} \Big) N_5 + N_5 \\
&= \sum_{j=1}^{4} \underline{N_j} + \cancel{-N_5} + \cancel{N_5} \\
&= \sum_{j=1}^{4} \underline{N_j}
\end{aligned}
$$

Because $\underline{N_j}$ were defined as the shape functions of the 4-noded quadrilateral element, their sum has been proven to be equal to 1. Hence, the condition is verified for the 5-noded quadrilateral element.