Universitat Politècnica de Catalunya

Computational Solid Mechanics and Dynamics

Master's Degree in Numerical Methods in Engineering

On 'Isoparametric Representation' and 'Structures of Revolution'

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1 Assignment 4.1

1.1 a), b)

If writing a relation between the coordinates of the nodes of an element and the Cartesian coordinates, it is obtained that

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 \tag{1}$$

Therefore the shape functions are chosen such that N_i must be a function which is unity at the nodes. Specifically, the shape functions corresponding to the depicted nodes are

$$\begin{cases} N_1 = \frac{\xi(\xi-1)}{2} \\ N_2 = \frac{\xi(\xi+1)}{2} \\ N_3 = -(\xi-1)(\xi+1) \end{cases} \tag{2}$$

Then, using the nodal coordinates in the local system

$$x = \frac{\xi(\xi+1)}{2} l - (\xi-1)(\xi+1)(\frac{l}{2} + \alpha l) = \frac{l}{2}(1+\xi)(2\alpha - 2\xi\alpha + 1) \tag{3}$$

It is possible to see from equation (3) that the coordinates of the nodes are obtained if the value of ξ are substituted. Now the derivative of (3) with respect to ξ is easily calculated.

$$\frac{dx}{d\xi} = \frac{l}{2}(1 - 4\alpha\xi) \tag{4}$$

Equation (4) is clearly the Jacobian. It is to be noted that $J = l/2$ when $\alpha = 0$. On the other hand, to check weather the value of the Jacobian is positive on the whole domain when $1/4 < \alpha < 1/4$, equation (4) may be plotted for the extreme cases $\xi = 1, -1$ as a function of alpha. Figure 1 shows that the jacobian is always positive for this range of α, being it null at the extreme values.

The strain displacement matrix is simply calculated as

$$\mathbf{B} = J^{-1}\frac{d\mathbf{N}}{d\xi} = \frac{2}{1 - 4\alpha\xi}\left[\xi - 1/2, \quad \xi + 1/2, \quad -2\xi\right] \tag{5}$$

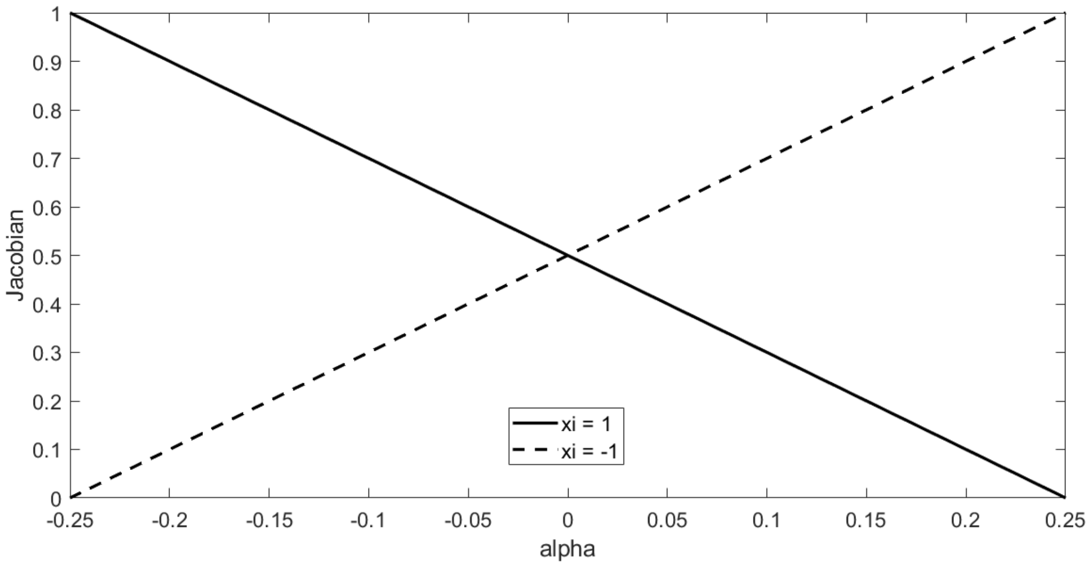

Figure 1: Variation of the Jacobian at the extreme points for various values of α.

2 Assignment 4.2

2.1 a)

The stiffness matrix of an element which is revolved from its original form can be calculated taking into account that the volume integral can be simply translated into a surface integral by considering the whole ring of revolution. In such a case,

$$\mathbf{K_{ij}} = 2\pi \int \mathbf{B_i^T D B_j} r dr dz \tag{6}$$

Where $\mathbf{D}$ is the given strain-stress matrix for an isotropic material with zero Poisson ratio. The $\mathbf{B}$ matrix is calculated as

$$\mathbf{B} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cc} 1 & \quad 2 \end{array} \\ \begin{pmatrix} \partial N_i/\partial r & 0 \\ 0 & \partial N_i/\partial z \\ \mathrm{N}_i/r & 0 \\ \partial N_i/\partial z & \partial N_i/\partial r \end{pmatrix} \end{array}$$

Now, the shape functions of the triangle are determined based on the coordinates we are given. The plot of the nodal coordinates is shown in Fig. 2.

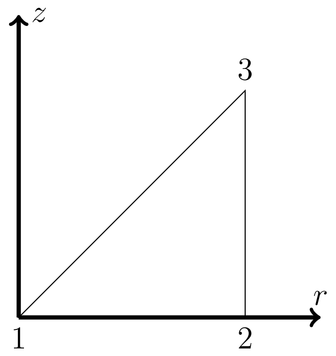

Figure 2: Local numbering of the element.

Of course as a condition for completeness the shape functions must sum one at each node, and zero at the others. The shape functions are described as

$$N_i = \frac{a_i + b_i r + c_i z}{2\Delta} \tag{7}$$

Where (in cyclic order, ijm)

$$\begin{cases} a_i = r_j z_m - r_m z_j \\ b_i = z_j - z_m \\ c_i = r_m - r_j \end{cases} \tag{8}$$

When computed, taking into account that $\Delta = ab/2$,

$$\begin{cases} N_1 = 1 - r/a \\ N_3 = z/b \\ N_2 = 1 - N_1 - N_3 = r/a - z/b \end{cases} \tag{9}$$

The parameters r, z can be expressed in terms of the shape functions by the following relation.

$$\begin{cases} r = N_1 r_1 + N_2 r_2 + N_3 r_3 = r - \frac{za}{b} + \frac{za}{b} = r \\ z = N_1 z_1 + N_2 z_2 + N_3 z_3 = z \end{cases} \tag{10}$$

When the derivatives are done with respect to the variables of each shape function the following matrix is obtained

$$
\mathbf{B} = \begin{array}{c c}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix}
-1/a & 0 & 1/a & 0 & 0 & 0 \\
0 & 0 & 0 & -1/b & 0 & 1/b \\
1/r - 1/a & 0 & 1/a - z/(br) & 0 & z/(br) & 0 \\
0 & -1/a & -1/b & 1/a & 1/b & 0
\end{pmatrix}
\end{array}
$$

Now, the simplest numerical integration procedure is to compute all quantities at the centroid point ($\bar{r} = 2a/3, \quad \bar{z} = b/3$).

$$
\mathbf{K_{ij}} = 2\pi \mathbf{B_i^T D B_j} \bar{r} \Delta \tag{11}
$$

Where $\mathbf{B}$ is the value of the stain-displacement matrix at the centroid point. The complete matrix will thus be

$$
\mathbf{K} = 2\pi \mathbf{B^T D B} \bar{r} \Delta \tag{12}
$$

When doing so with Matlab (code attached) the following matrix is obtained

$$
\mathbf{K} = E\pi \begin{array}{c c}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 4 \\ 6 \end{array} & \begin{pmatrix}
\text{5b/6} & 0 & \text{-b/2} & 0 & \text{b/6} & 0 \\
 & \text{b/3} & \text{a/3} & \text{-b/3} & \text{-a/3} & 0 \\
 & & \text{b/3+ 2a}^2/(3b) & \text{-a/3} & \text{b/3- 2a}^2/(3b) & 0 \\
 & & & \text{b/3+ 2a}^2/(3b) & \text{a/3} & \text{-2a}^2/(3b) \\
Symm. & & & & \text{b/3+ 2a}^2/(3b) & 0 \\
 & & & & & \text{2a}^2/(3b)
\end{pmatrix}
\end{array}
$$

2.2 b)

To check that the sum of rows and columns 2,4,6 vanish but not the 1,3,5, little operations are added to the code that indeed confirm that rows and columns 2,4,6 sum zero but rows and columns 1,3,5 sum non-zero. The reason for this is that there is symmetry about the z axis, and therefore the stresses are independent of the θ coordinate, which means that the forces on the component z have to be in equilibrium, but not the r component (since there is symmetry). This explains the vanishing of these rows pertaining to the z component.

$$\sum_{columns} = \sum_{rows} = [\frac{E\pi b}{2} \quad 0 \quad \frac{E\pi b}{2} \quad 0 \quad \frac{E\pi b}{2} \quad 0] \tag{13}$$

2.3 c)

The distributed body forces can be found as

$$\mathbf{f} = \int_S \mathbf{N_s^T} \mathbf{b} dS \tag{14}$$

Where $\mathbf{N_s}$ denotes the shape function matrix evaluated along the surface where the surface traction acts. Integrating the equations explicitly along for the three surfaces the total distribution of body force is obtained [1].

$$\mathbf{f} = \frac{2\pi\bar{r}\Delta}{3}[b_r \quad b_z \quad b_r \quad b_z \quad b_r \quad b_z]^T = \frac{2\pi a^2 b}{9}[0 \quad -g \quad 0 \quad -g \quad 0 \quad -g]^T = \tag{15}$$

A Appendix:Code

```
syms a b r z

E = [1 0 0 0; 0 1 0 0; 0 0 1 0; 0 0 0 1/2];

B = [-1/a 0 1/a 0 0 0; ...
    0 0 0 -1/b 0 1/b; ...
    1/r-1/a 0 1/a-z/(r*b) 0 z/(r*b) 0; ...
    0 -1/a -1/b 1/a 1/b 0];

B = subs(B,r,2*a/3);
B = subs(B,z,b/3);

A = E*B;

Bt = transpose(B);

K = Bt*A;
K = K*b*a*2*a/3;
%Sum of rows and columns 2,4,6
sum(K(:,6))
```

```
sum(K(:,4))
sum(K(:,2))
sum(K(2,:))
sum(K(4,:))
sum(K(6,:))

%Sum of rows and columns 1,3,5
sum(K(:,1))
sum(K(:,3))
sum(K(:,5))
sum(K(1,:))
sum(K(3,:))
sum(K(5,:))
```

References

[1] O.C. Zienkiewics and K. Morgan, *Finite Elements and Approximation*. Dover books ISNB 0-486-4530149 (1983).