Computational Structural Mechanics and Dynamics

Master of Science in Computational Mechanics
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Homework 4

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Assignment 4.1

1. Compute the entries of K^e for the following axisymmetric triangle:

$$r_1 = 0 \qquad r_2 = r_3 = a \qquad z_1 = z_2 = 0 \qquad z_3 = b$$

The material is isotropic with $\nu = 0$ for which the stress-strain matrix is,

$$\mathbf{E} = E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

According with the element, the shape functions in natural coordinates for a triangular elements are $N_1 = 1 - \xi - \eta$, $N_2 = \xi$,$N_3 = \eta$.

$$\boldsymbol{B} = \boldsymbol{DN} \qquad \boldsymbol{B} = \begin{bmatrix} \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial r} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \frac{\partial N_3}{\partial z} \\ \frac{N_1}{r} & \frac{N_2}{r} & \frac{N_3}{r} & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \frac{\partial N_3}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial r} \end{bmatrix}$$

In order to compute $\mathbf{B}$ matrix, the differentiation of shape functions respect to cylindrical coordinates are required by isoparametric interpolation. On the other side r should be interpolated from the nodal coordinates $r = a(\xi + \eta)$

$$\frac{\partial N_1}{\partial r} = \frac{-1}{a} \qquad \frac{\partial N_2}{\partial r} = \frac{1}{a} \qquad \frac{\partial N_3}{\partial r} = 0 \qquad \frac{\partial N_1}{\partial z} = 0 \qquad \frac{\partial N_2}{\partial z} = \frac{-1}{b} \qquad \frac{\partial N_3}{\partial z} = \frac{1}{b}$$

$$\boldsymbol{B} = \begin{bmatrix} \frac{-1}{a} & \frac{1}{a} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{b} & \frac{1}{b} \\ \frac{1-\xi-\eta}{a(\xi+\eta)} & \frac{\xi}{a(\xi+\eta)} & \frac{\eta}{a(\xi+\eta)} & 0 & 0 & 0 \\ 0 & \frac{-1}{b} & \frac{1}{b} & \frac{-1}{a} & \frac{1}{a} & 0 \end{bmatrix}$$

The element stiffness equation and the consistent nodal force are:

$$K^{(e)} = 2\pi \int_{\Omega^e} \mathbf{B^T E\, B}\, r\, d\Omega \qquad \mathbf{f^e} = 2\pi \int_A \mathbf{N^t}\, \mathbf{b}\, r\, dA$$

With Jacobian and its determinant as:

$$J = \sum_{i=1}^{n} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} r_i & \frac{\partial N_i}{\partial \xi} z_i \\ \frac{\partial N_i}{\partial \eta} r_i & \frac{\partial N_i}{\partial \eta} z_i \end{bmatrix} = \begin{bmatrix} a & 0 \\ a & b \end{bmatrix} \qquad det(J) = ab$$

Using Gauss centroid rule to compute the integration, we have to substitute $\xi_1 = \xi_2 = \xi_3 = \frac{1}{3}$ and $\omega_k = \omega_l = 0.5$ into the element stiffness matrix. Therefore we will obtain K as the following:

$$\boldsymbol{K} = \pi\, ab\, E \begin{bmatrix} \frac{5}{6a} & -\frac{1}{2a} & \frac{1}{6a} & 0 & 0 & 0 \\ -\frac{1}{2a} & \frac{2a}{3}((\frac{5}{4a^2}) + \frac{1}{2b^2}) & \frac{2a}{3}((\frac{1}{4a^2}) - \frac{1}{2b^2}) & \frac{1}{3b} & -\frac{1}{3b} & 0 \\ \frac{1}{6a} & \frac{2a}{3}((\frac{1}{4a^2}) - \frac{1}{2b^2}) & \frac{2a}{3}((\frac{1}{4a^2}) + \frac{1}{2b^2}) & -\frac{1}{3b} & \frac{1}{3b} & 0 \\ 0 & \frac{1}{3b} & -\frac{1}{3b} & \frac{1}{3a} & -\frac{1}{3a} & 0 \\ 0 & -\frac{1}{3b} & \frac{1}{3b} & -\frac{1}{3a} & \frac{2a}{3}((\frac{1}{2a^2}) + \frac{1}{b^2}) & -\frac{2a}{3b^2} \\ 0 & 0 & 0 & 0 & -\frac{2a}{3b^2} & \frac{2a}{3b^2} \end{bmatrix}$$

2. Show that the sum of the rows (and columns) 2, 4 and 6 of $K^{(e)}$ must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.

   In our case, due to the formulation of matrix **B**, the stiffness matrix **K** has the sum in the second, fourth and sixth rows and columns equal to zero. The physical meaning, rise from the zero energy due to the free rigid body motion on z direction (axisymmetric).

3. Compute the consistent force vector $f^{(e)}$ for gravity forces $b = [0, -g]^T$. Where the shape functions matrix **N** is defined as:

$$\mathbf{N} = \begin{bmatrix} N_1^{(e)} & N_2^{(e)} & N_3^{(e)} & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1^{(e)} & N_2^{(e)} & N_3^{(e)} \end{bmatrix} \Longrightarrow N_1 = 1 - \xi - \eta \quad N_2 = \xi \quad N_3 = \eta$$

   Using Gauss centroid rule, we have:

$$f = \pi\, ab\, E \begin{bmatrix} 0 & 0 & 0 & -\frac{2g\,a}{9} & -\frac{2g\,a}{9} & -\frac{2g\,a}{9} \end{bmatrix}^T$$