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Assignment 4.1 CSMD

Part - 1

$r_1 = 0 \;,\; r_2 = r_3 = a \;,\; z_1 = z_2 = 0 \;,\; z_3 = b$

where $\nu = 0$

Strain-Stress matrix $E = E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$ $\qquad K^e = ?$

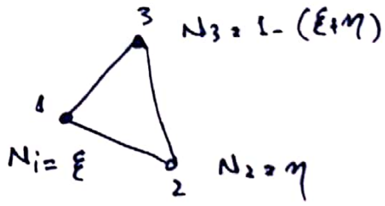
Stiffness matrix can be computed by Numerical Integration by Gauss Rules.

$$K^e = \int_{-1}^{1}\int_{-1}^{1} h\, B^T E B \det J \, d\xi d\eta \approx \sum_{i=1}^{p_1}\sum_{j=1}^{p_2} w_i w_j F(\xi,\eta)$$

In a more elevative way for "axissymm triangle"

$$K^e = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j B^T(\xi_i,\eta_j) E B(\xi_i,\eta_j)\, r(\xi_i,\eta_j)\, J(\xi_i,\eta_j) \longrightarrow \text{(A)}$$

where Shape function $\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \\ 1-(\xi+\eta) \end{bmatrix}$



And

$$B = DN = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & N_3^e & 0 \\ 0 & N_1^e & 0 & N_2^e & 0 & N_3^e \end{bmatrix} \quad \because \begin{bmatrix} N_i^e \\ N_i^e \end{bmatrix} = \begin{bmatrix} r \\ z \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} \end{bmatrix}$$

And

$$J = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix}$$

From triangular shape function, we have

$$r = N_1 r_1 + N_2 r_2 + N_3 r \quad \& \quad z = N_1 z_1 + N_2 z_2 + N_3 z_3$$

$$= \xi(0) + \eta(a) + (1-(\xi+\eta))a \qquad = \xi(0) + \eta(0) + (1-(\xi+\eta))(b)$$

$$\boxed{r = a - a\xi} \qquad \boxed{z = b - b\xi - b\eta}$$

$$\frac{\partial r}{\partial \xi} = -a \quad \& \quad \frac{\partial r}{\partial \eta} = 0 \quad \& \quad \frac{\partial z}{\partial \xi} = -b \quad \& \quad \frac{\partial z}{\partial \eta} = -b \rightarrow (1)$$

So,

$$J = \begin{bmatrix} -a & -b \\ 0 & -b \end{bmatrix} \text{ and } J^{-1} = \begin{bmatrix} -\frac{1}{a} & \frac{1}{a} \\ 0 & -\frac{1}{b} \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial r} & \frac{\partial \eta}{\partial r} \\ \frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} \end{bmatrix}$$

$$\frac{\partial \xi}{\partial r} = -\frac{1}{a} \quad \& \quad \frac{\partial \eta}{\partial r} = \frac{1}{a} \quad \& \quad \frac{\partial \xi}{\partial z} = 0 \quad \& \quad \frac{\partial \eta}{\partial z} = -\frac{1}{b} \rightarrow (2)$$

As,

$$\left.\begin{array}{l} N_1 = \xi \Rightarrow \frac{\partial N_1}{\partial \xi} = 1 \quad \& \quad \frac{\partial N_1}{\partial \eta} = 0 \\ N_2 = \eta \Rightarrow \frac{\partial N_2}{\partial \xi} = 0 \quad \& \quad \frac{\partial N_2}{\partial \eta} = 1 \\ N_3 = 1-(\xi+\eta) \Rightarrow \frac{\partial N_3}{\partial \xi} = -1 \quad \& \quad \frac{\partial N_3}{\partial \eta} = -1 \end{array}\right\} \longrightarrow (3)$$

So, use the values from Eq (1), (2) & (3) to calculate the derivatives of shape function.

$$\frac{\partial N_1}{\partial r} = \frac{\partial N_1}{\partial \xi}\cdot\frac{\partial \xi}{\partial r} + \frac{\partial N_1}{\partial \eta}\cdot\frac{\partial \eta}{\partial r} = (1)\left(-\frac{1}{a}\right) + 0\left(\frac{1}{a}\right) \Rightarrow \boxed{\frac{\partial N_1}{\partial r} = -\frac{1}{a}}$$

$$\frac{\partial N_1}{\partial z} = \frac{\partial N_1}{\partial \xi}\cdot\frac{\partial \xi}{\partial z} + \frac{\partial N_1}{\partial \eta}\cdot\frac{\partial \eta}{\partial z} = (1)(0) + 0\left(-\frac{1}{b}\right) \Rightarrow \boxed{\frac{\partial N_1}{\partial z} = 0}$$

$$\frac{\partial N_2}{\partial r} = \frac{\partial N_2}{\partial \xi}\cdot\frac{\partial \xi}{\partial r} + \frac{\partial N_2}{\partial \eta}\cdot\frac{\partial \eta}{\partial r} = (0)\left(-\frac{1}{a}\right) + (1)\left(\frac{1}{a}\right) \Rightarrow \boxed{\frac{\partial N_2}{\partial r} = \frac{1}{a}} \quad (3)$$

$$\frac{\partial N_2}{\partial z} = \frac{\partial N_2}{\partial \xi}\cdot\frac{\partial \xi}{\partial z} + \frac{\partial N_2}{\partial \eta}\cdot\frac{\partial \eta}{\partial z} = (0)(0) + (1)\left(-\frac{1}{b}\right) \Rightarrow \boxed{\frac{\partial N_2}{\partial z} = -\frac{1}{b}}$$

$$\frac{\partial N_3}{\partial r} = \frac{\partial N_3}{\partial \xi}\cdot\frac{\partial \xi}{\partial r} + \frac{\partial N_2}{\partial \eta}\cdot\frac{\partial \eta}{\partial r} = (-1)\left(-\frac{1}{a}\right) + (-1)\left(\frac{1}{a}\right) \Rightarrow \boxed{\frac{\partial N_3}{\partial r} = 0}$$

$$\frac{\partial N_3}{\partial z} = \frac{\partial N_3}{\partial \xi}\cdot\frac{\partial \xi}{\partial z} + \frac{\partial N_3}{\partial \eta}\cdot\frac{\partial \eta}{\partial z} = (-1)(0) + (-1)\left(-\frac{1}{b}\right) \Rightarrow \boxed{\frac{\partial N_3}{\partial z} = \frac{1}{b}}$$

And

$$\boxed{\frac{N_1}{r} = \frac{\xi}{a-a\xi}}\ , \quad \boxed{\frac{N_2}{r} = \frac{\eta}{(a-a\xi)}}\ , \quad \boxed{\frac{N_3}{r} = \frac{1-(\xi+\eta)}{a-a\xi}}$$

Put all the entities into B matrix.

$$B = B(\xi_i,\eta_i) = \begin{bmatrix} -1/a & 0 & 1/a & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/b & 0 & 1/b \\ \frac{\xi}{a-a\xi} & 0 & \frac{\eta}{a-a\xi} & 0 & \frac{1-(\xi+\eta)}{a-a\xi} & 0 \\ 0 & -1/a & -1/b & 1/a & 1/b & 0 \end{bmatrix}$$

And

$$B^T E B = E \begin{bmatrix} -1/a & 0 & \xi/a-a\xi & 0 \\ 0 & 0 & 0 & -1/a \\ 1/a & 0 & \eta/a-a\xi & -1/b \\ 0 & -1/b & 0 & 1/a \\ 0 & 0 & \frac{1-(\xi+\eta)}{a-a\xi} & 1/b \\ 0 & 1/b & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} -1/a & 0 & 1/a & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/b & 0 & 1/b \\ \frac{\xi}{a-a\xi} & 0 & \frac{\eta}{a-a\xi} & 0 & \frac{1-(\xi+\eta)}{a-a\xi} & 0 \\ 0 & -1/a & -1/b & 1/a & 1/b & 0 \end{bmatrix}}_{\Downarrow}$$

$$\begin{bmatrix} -1/a & 0 & 1/a & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/b & 0 & 1/b \\ \xi/a-a\xi & 0 & \eta/a-a\xi & 0 & \frac{1-(\xi+\eta)}{a-a\xi} & 0 \\ 0 & -1/2a & -\frac{1}{2b} & 1/2a & 1/2b & 0 \end{bmatrix}$$

(4)

$$B^T E B = E \begin{bmatrix} \frac{1}{a^2}+\frac{\xi^2}{(a-a\xi)^2} & 0 & -\frac{1}{a^2}+\frac{\xi\eta}{(a-a\xi)^2} & 0 & \frac{\xi(1-(\xi+\eta))}{(a-a\xi)^2} & 0 \\ 0 & \frac{1}{2a^2} & \frac{1}{2ab} & -\frac{1}{2a^2} & -\frac{1}{2ab} & 0 \\ -\frac{1}{a^2}+\frac{\xi\eta}{(a-a\xi)^2} & \frac{1}{2ab} & \frac{1}{a^2}+\frac{\eta^2}{(a-a\xi)^2}+\frac{1}{2b^2} & \frac{-1}{2ab} & \frac{\eta(1-(\xi+\eta))}{(a-a\xi)^2}-\frac{1}{2b^2} & 0 \\ 0 & -\frac{1}{2a^2} & -\frac{1}{2ab} & \frac{1}{b^2}+\frac{1}{2a^2} & \frac{1}{2ab} & -\frac{1}{b^2} \\ \frac{\xi(1-(\xi+\eta))}{(a-a\xi)^2} & -\frac{1}{2ab} & \frac{\eta(1-(\xi+\eta))}{(a-a\xi)^2}-\frac{1}{2b^2} & \frac{1}{2ab} & \frac{(1-(\xi+\eta))^2}{(a-a\xi)^2}+\frac{1}{2b^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{b^2} & 0 & \frac{1}{b^2} \end{bmatrix}$$

Put in Eq (A) and Applying Gauss Quadrature (1 point) with $\xi = \frac{1}{3}$, $\eta = \frac{1}{3}$

and $\det J = |J| = ab$ and $w_i = w_j = 1$

So k^e matrix will becomes

$$K^e = \frac{2}{3} E a^2 b \begin{bmatrix} 1/a^2 + 1/4a^2 & 0 & -1/a^2 + \frac{1}{4a^2} & 0 & \frac{1}{4a^2} & 0 \\ 0 & \frac{1}{2a^2} & 1/2ab & -1/2a^2 & -1/2ab & 0 \\ -1/a^2 + 1/4a^2 & 1/2ab & 1/a^2 + 1/4a^2 + \frac{1}{2b^2} & -1/2ab & 1/4a^2 - 1/2b^2 & 0 \\ 0 & -1/2a^2 & -1/2ab & 1/b^2 + 1/2a^2 & 1/2ab & -1/b^2 \\ 1/4a^2 & -1/2ab & 1/4a^2 - 1/2b^2 & 1/2ab & 1/4a^2 + 1/2b^2 & 0 \\ 0 & 0 & 0 & -1/b^2 & 0 & 1/b^2 \end{bmatrix}$$

Part - 2

$$R\text{-}2 = \frac{1}{2a^2} + \frac{1}{2ab} - \frac{1}{2a^2} - \frac{1}{2ab} = 0$$ ⑤

$$R\text{-}4 = -\frac{1}{2a^2} - \frac{1}{2ab} + \frac{1}{b^2} + \frac{1}{2a^2} + \frac{1}{2ab} - \frac{1}{b^2} = 0$$

$$R\text{-}6 = -\frac{1}{b^2} + \frac{1}{b^2} = 0$$

&

$$C\text{-}2 = +\frac{1}{2a^2} + \frac{1}{2ab} - \frac{1}{2a^2} - \frac{1}{2ab} = 0$$

$$C\text{-}4 = -\frac{1}{2a^2} - \frac{1}{2ab} + \frac{1}{b^2} + \frac{1}{2a^2} + \frac{1}{2ab} - \frac{1}{b^2} = 0$$

$$C\text{-}6 = -\frac{1}{b^2} + \frac{1}{b^2} = 0$$

Now

$$R\text{-}1 = \frac{1}{a^2} + \frac{1}{4a^2} - \frac{1}{a^2} + \frac{1}{4a^2} + \frac{1}{4a^2} \neq 0$$

$$R\text{-}3 = -\frac{1}{a^2} + \frac{1}{4a^2} + \frac{1}{2ab} + \frac{1}{a^2} + \frac{1}{4a^2} + \frac{1}{2b^2} - \frac{1}{2ab} + \frac{1}{4a^2} - \frac{1}{2b^2} \neq 0$$

$$R\text{-}5 = \frac{1}{4a^2} - \frac{1}{2ab} + \frac{1}{4a^2} - \frac{1}{2b^2} + \frac{1}{2ab} + \frac{1}{4a^2} + \frac{1}{2b^2} \neq 0$$

&

$$C\text{-}1 = \frac{1}{a^2} + \frac{1}{4a^2} - \frac{1}{a^2} + \frac{1}{4a^2} + \frac{1}{4a^2} \neq 0$$

$$C\text{-}3 = -\frac{1}{a^2} + \frac{1}{4a^2} + \frac{1}{2ab} + \frac{1}{a^2} + \frac{1}{4a^2} + \frac{1}{2b^2} - \frac{1}{2ab} + \frac{1}{4a^2} - \frac{1}{2b^2} \neq 0$$

$$C\text{-}5 = \frac{1}{4a^2} - \frac{1}{2ab} + \frac{1}{4a^2} - \frac{1}{2b^2} + \frac{1}{2ab} + \frac{1}{4a^2} + \frac{1}{2b^2} \neq 0$$

Comments :-

It is cleared that rows and colums which contain $q_r \left(\frac{N_i}{r}\right)$ entities are not summed up to zero. The extra term from axis symmetry create near singularity and make calculation complex. while the rows and column which do not share 'q_r' entities are summed up to zero.

part-3

As Consistent force vector is

$$f^e = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j N^T(\xi_i,\eta_j)\, b(\xi_i,\eta_j)\, r(\xi_i,\eta_j) \det J(\xi_i,\eta_j)$$

As $b = \begin{bmatrix} 0 \\ -q \end{bmatrix}$, $N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$

$$N^T = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} = \begin{bmatrix} \xi & 0 \\ 0 & \xi \\ \eta & 0 \\ 0 & \eta \\ 1-(\xi+\eta) & 0 \\ 0 & 1-(\xi+\eta) \end{bmatrix}$$

take $w_i w_j = 1$ and $\xi = \frac{1}{3} = \eta$ and $|J| = ab$

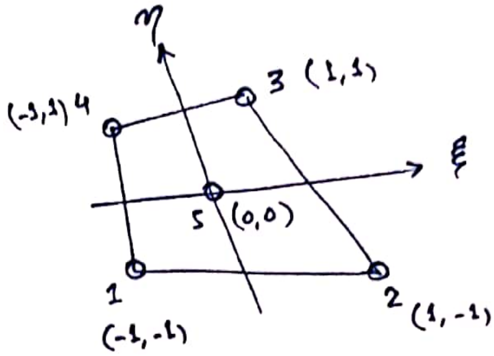
$$r = \frac{2}{3}a$$

$$f^e = \frac{2}{3}a^2 b \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \\ 1/3 & 0 \\ 0 & 1/3 \\ 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 \\ -q \end{bmatrix} = \frac{2}{3}a^2 b \begin{bmatrix} 0 \\ -q/3 \\ 0 \\ -q/3 \\ 0 \\ -q/3 \end{bmatrix}$$

Assignment. No. 4.2

Find Shape function of Quadrilateral

N_1, N_2, N_3, N_4, N_5



Shape Function for $i = 1,2,3,4$ can be calculated by $N_i = N_i + \alpha N_5$ while N_5 through line-product method.

So

$$N_5 = \alpha\, L_{1-2} L_{2-3} L_{3-4} L_{4-1}$$

$$N_5 = \alpha\, (1+\eta)(1-\xi)(1-\eta)(1+\xi)$$

$$N_5 = \alpha\, (1-\eta^2)(1-\xi^2)$$

$$N_5 = \alpha\, (1-\xi^2)(1-\eta^2)$$

For $\alpha = 1$ at node 5

$N_5 = 1$ and $N_1 = N_2 = N_3 = N_4 = 0$

For Corner nodes 1, 2, 3, 4

$$N_1 = C\,(1-\xi)(1-\eta)(\xi)(\eta)$$

$$N_2 = C\,(1+\xi)(1-\eta)(\xi)(\eta)$$

$$N_3 = C\,(1+\xi)(1+\eta)(\xi)(\eta)$$

$$N_4 = C\,(1-\xi)(1+\eta)(\xi)(\eta)$$

For $C = 1/4$ at $i = 1,2,3,4$

$N_1 = N_2 = N_3 = N_4 = 1$ and $N_5 = 0$

$i =$ 1 2 3 4