Master's Degree Numerical Methods in Engineering

Universitat Politècnica de Catalunya
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Computational Structural Mechanics and Dynamics

Homework 4: Isoparametric representation

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Assignment 4.1

A 3-node straight bar element is defined by 3 nodes: 1, 2 and 3 with axial coordinates $x_1,\ x_2$ and x_3 respectively as illustrated in figure below. The element has axial rigidity EA, and length $L = x_1 - x_2$. The axial displacement is $u(x)$. The 3 degrees of freedom are the axial node displacement $u_1,\ u_2$ and u_3. The isoparametric definition of the element is

$$\begin{bmatrix} 1 \\ x \\ u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \cdot \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

in which $N_1^e(\xi)$ are the shape functions of a three bar element. Node 3 lies between 1 and 2 but is not necessarily at the midpoint $x = L/2$. For convenience define,

$$x_1 = 0,\ x_2 = L,\ x_3 = (1/2 + \alpha) \cdot L$$

where $-1/2 < \alpha < 1/2$ characterizes the location of node 3 with respect to the element center. If $\alpha = 0$ node 3 is located at the midpoint between 1 and 2.

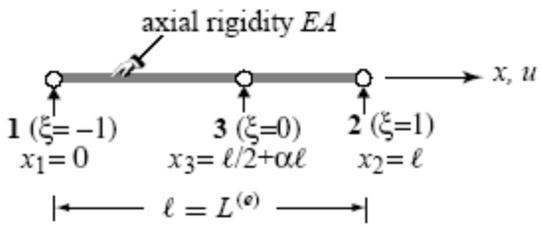

Figure 1: The three-node bar element in its local system

1. From the above definitions get the Jacobian $J = dx/d\xi$ in terms of L, α and ξ. Show that:
   - if -1/4 < α < 1/4 then $J > 0$ over the whole element -1<ξ<1.
   - if $\alpha = 0$, $J = 1/2$ is a constant over the element.
2. Obtain the 1x3 strain displacement matrix B relating $e = du/dx = B \cdot u^e$ where u^e is the column 3-vector of the node displacement u_1, u_2 and u_3. The entries of B are functions of L, α and ξ

Hint: $\mathbf{B} = dN/dx = J^{-1} \cdot dN/d\xi$, where $N = [N_1,\ N_2,\ N_3]$ and J comes from item a).

Assignment 4.2

1. Compute the entries of $\mathbf{K}^e$ for the following axisymmetric triangle:

$$r_1 = 0,\ r_2 = r_3 = a,\ z_1 = z_2 = 0,\ z_3 = b$$

The material is isotropic with $\nu = 0$ for which the stress-strain matrix is,

$$E = E \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

2. Show that the sum of the rows (and columns) 2, 4 and 6 of K^e must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.
3. Compute the consistent force vector f_e for gravity forces $b = [0, -g]^T$.

Resolution

1 Assignment 4.1: First task

The goal of this task is to find the jacobian of the transformation. Therefore, in the following equations the definitions of the shape functions in terms of the ξ parameter are shown:

$$N_1(\xi) = \frac{(\xi_2 - \xi) \cdot (\xi_3 - \xi)}{(\xi_2 - \xi_1) \cdot (\xi_3 - \xi_1)} = \frac{(1 - \xi) \cdot (-\xi)}{2} = \frac{\xi^2 - \xi}{2}$$

$$N_2(\xi) = \frac{(\xi_3 - \xi) \cdot (\xi_1 - \xi)}{(\xi_3 - \xi_2) \cdot (\xi_1 - \xi_2)} = \frac{-\xi \cdot (-1 - \xi)}{2} = \frac{\xi^2 + \xi}{2}$$

$$N_3(\xi) = \frac{(\xi_1 - \xi) \cdot (\xi_3 - \xi)}{(\xi_1 - \xi_3) \cdot (\xi_2 - \xi_3)} = \frac{(-1 - \xi) \cdot (1 - \xi)}{-1} = 1 - \xi^2$$

where $\xi_1 = -1$, $\xi_3 = 0$ and $\xi_2 = 1$.

Once the shape functions in terms of the parameter ξ are found, the geometric variable x is described in terms of the parameter ξ, leading to an ISO-parametric description:

$$x(\xi) = x_1 \cdot N_1(\xi) + x_2 \cdot N_2(\xi) + x_3 \cdot N_3(\xi)$$

remembering $x_1 = 0$, $x_2 = L$ and $x_3 = (\frac{1}{2} + \alpha) \cdot L$:

$$x(\xi) = L \cdot \frac{\xi^2 + \xi}{2} + (\frac{1}{2} + \alpha) \cdot L \cdot (1 - \xi^2)$$

Now it is possible to calculate the jacobian of the transformation:

$$J = \frac{dx}{d\xi} = \frac{L}{2} \cdot (2\xi + 1) + (\frac{1}{2} + \alpha) \cdot L \cdot (-2\xi) = L \cdot \left(\frac{1}{2} - 2\xi\alpha\right) \tag{1}$$

As written in the assignment, if $-1/4 < \alpha < 1/4$ the Jacobian is bigger than zero over the whole element (as the coordinate ξ varies from [-1;1] it does not modify the magnitude of the substracting term), see Equation (1).

If $\alpha = 0$ the Equation (1) is constant equal to $J = L/2$.

2 Assignment 4.1: Second task

To calculate the strain displacement matrix B, the derivatives of the shape functions with respect to the parameter ξ are needed and the inverse of the Jacobian is also used given to the fact described in Equation (2):

$$e = \frac{du}{dx} = \frac{d}{dx} \sum_{i=1}^{n=3} N_i(\xi) \cdot u_i = \sum_{i=1}^{n=3} \frac{N_i(\xi)}{d\xi} \cdot \frac{d\xi}{dx} \cdot u_i = \sum_{i=1}^{n=3} \frac{N_i(\xi)}{d\xi} \cdot J^{-1}(\xi) \cdot u_i \tag{2}$$

$$e = \begin{bmatrix} J^{-1}(\xi_1) \cdot \frac{dN_1}{d\xi} & J^{-1}(\xi_2) \cdot \frac{dN_2}{d\xi} & J^{-1}(\xi_3) \cdot \frac{dN_3}{d\xi} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{2}{L \cdot (1+4\alpha)} \cdot \frac{2\xi - 1}{2} & \frac{2}{L \cdot (1-4\alpha)} \cdot \frac{2\xi + 1}{2} & \frac{-4\xi}{L} \end{bmatrix}$$

3 Assignment 4.2: First task

Compute the entries of a $\boldsymbol{K}^e$ for an axisymmetric triangle.

To define the axisymmetric triangle it is a good idea to make a change of coordinates using triangular coordinates:

$$\begin{bmatrix} 1 \\ r \\ z \\ u_r \\ u_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & a & a \\ 0 & 0 & b \\ u_{r1} & u_{r2} & u_{r3} \\ u_{z1} & u_{z2} & u_{z3} \end{bmatrix} \cdot \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

The triangular coordinates are calculated as:

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{2A} \cdot \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ r \\ z \end{bmatrix} = \frac{1}{ab} \cdot \begin{bmatrix} ab - a0 & 0 - b & a - a \\ a0 - 0b & b - 0 & 0 - a \\ 00 - a0 & 0 - 0 & a - 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ r \\ z \end{bmatrix}$$

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{ab} \cdot \begin{bmatrix} ab & -b & 0 \\ 0 & b & -a \\ 0 & 0 & a \end{bmatrix} \cdot \begin{bmatrix} 1 \\ r \\ z \end{bmatrix}$$

Then the shape functions are:

$$\zeta_1 = \frac{1}{ab} \cdot [ab - b \cdot r] = 1 - \frac{r}{a}$$
$$\zeta_2 = \frac{1}{ab} \cdot [b \cdot r - a \cdot z] = \frac{r}{a} - \frac{z}{b}$$
$$\zeta_3 = \frac{1}{ab} \cdot [a \cdot z] = \frac{z}{b}$$

The strain vector according to Zienkiewicz *chapter 5: Axisymmetric stress analysis* is:

$$\varepsilon = \begin{bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{bmatrix} = \begin{bmatrix} \partial u_r/\partial r \\ \partial u_z/\partial z \\ u_r/r \\ \partial u_r/\partial z + \partial u_z/\partial r \end{bmatrix} \cdot \underline{u}_i = [B] \cdot \begin{bmatrix} u_{r1} \\ u_{z1} \\ u_{r2} \\ u_{z2} \\ u_{r3} \\ u_{z3} \end{bmatrix}$$

Then the **B** matrix is calculated as:

$$B = \begin{bmatrix} \partial N_i/\partial r & 0 \\ 0 & \partial N_i/\partial z \\ N_i/r & 0 \\ \partial N_i/\partial z & \partial N_i/\partial r \end{bmatrix} = \begin{bmatrix} -1/a & 0 & 1/a & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/b & 0 & 1/b \\ (1/r - 1/a) & 0 & (1/a - z/r \cdot b) & 0 & z/r \cdot b & 0 \\ 0 & -1/a & -1/b & 1/a & 1/b & 0 \end{bmatrix}$$

To calculate the stiffness matrix the Equation (3) presents the conditions to have an axisymmetric element:

$$\mathbf{K}_{ij}^{(e)} = 2\pi \int_0^a \int_0^{b/a \cdot r} \mathbf{B}_i^T \mathbf{D} \mathbf{B}_j \; r \; dz \; dr \tag{3}$$

Using MATLAB to integrate the stiffness matrix, the obtained result is:

$$K = 2\pi \cdot E \cdot \begin{bmatrix} \frac{2b}{3} & 0 & -\frac{b}{4} & 0 & \frac{b}{12} & 0 \\ & \frac{b}{6} & \frac{a}{6} & -\frac{b}{6} & -\frac{a}{6} & 0 \\ & & \frac{a^2}{6b} + \frac{4b}{9} & -\frac{a}{6} & -\frac{a^2}{6b} + \frac{b}{18} & 0 \\ & & & \frac{a^2}{3b} + \frac{b}{6} & \frac{a}{6} & -\frac{a^2}{3b} \\ & sym. & & & \frac{b}{9} + \frac{a^2}{6*b} & 0 \\ & & & & & \frac{a^2}{3*b} \end{bmatrix}$$

4 Assignment 4.2: Second task

The sum of the columns 2, 4 and 6 vanishes as these columns are multiplying the displacements u_{z1}, u_{z2} and u_{z3}. In this direction (z), the model can undergo a rigid body displacement that will not generate stresses. Only the relative movements between these nodes generate stresses, when all of them are equal, the generated force is zero. In the case of the columns 1, 3 and 5 the columns are multiplying the displacements u_{r1}, u_{r2} and u_{r3} and these coordinates cannot experiment a rigid body movement without generating stresses due to the nature of the model. The *axisymmetric* model, establishes a symmetry round an axis, and a displacement in the r direction is an anti-symmetric movement, therefore if there is a movement in this direction, it will generate forces (in the symmetric part of the model).

$$K(:,1) + K(:,3) + K(:,5) = \begin{bmatrix} b/2 \\ 0 \\ b/4 \\ 0 \\ b/4 \\ 0 \end{bmatrix}$$

$$K(:,2) + K(:,4) + K(:,6) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

5 Assignment 4.2: Third task

Compute the consistent force vector f_e for gravity forces $b = [0, -g]^T$.

$$f^{(e)} = 2\pi \cdot \int_{\Omega^{(e)}} (N^{(e)})^T \underline{b} \cdot r \; d\Omega$$

$$f^{(e)} = 2\pi \cdot \int_0^a \int_0^{b/a \cdot r} \begin{bmatrix} (1-\frac{r}{a}) & 0 \\ 0 & (1-\frac{r}{a}) \\ (\frac{rb-za}{ab}) & 0 \\ 0 & (\frac{rb-za}{ab}) \\ \frac{z}{b} & 0 \\ 0 & \frac{z}{b} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -g \end{bmatrix} \cdot r \; dzdr = -\pi \cdot g \cdot a^2 b \begin{bmatrix} 0 \\ \frac{1}{12} \\ 0 \\ \frac{3}{8} \\ 0 \\ \frac{1}{8} \end{bmatrix}$$

6 Conclusions

In the first exercise, the isoparametric description was used to describe a 1D bar with a moving node. Using the isoparametric description and the associated Jacobian of the transformation, the location of the node to have positive and negative Jacobians were determined. Remembering that having a negative Jacobian would mean to have a transformation that will modify the sign of the "fluxes". Finally, a triangular element with axisymmetry was studied, calculating the stiffness matrix, and the force vector. Also, it was confirmed that this kind of elements cannot undergo non-symmetric movements without registering stresses, as obtained in the sum of the columns. Finally, the force vector of body forces applied to the triangle was calculated.