Assignment 4

4.1.1 Compute the entries of K^e for the following axisymmetric triangle:

$$r_1 = 0, r_2 = r_3 = a, z_1 = z_2 = 0, z_3 = b$$

The material is isotropic with $\nu = 0$ for which the stress-strain matrix is,

$$E = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \tag{1}$$

The element stiffness matrix is given by the:

$$K^e = 2\pi \int_A rB^T EBdA \tag{2}$$

$$B_i = \frac{1}{2A^e} \begin{bmatrix} \frac{\delta N_i}{\delta r} & 0 \\ 0 & \frac{\delta N_i}{\delta z} \\ \frac{N_i}{r} & 0 \\ \frac{\delta N_i}{\delta z} & \frac{\delta N_i}{\delta r} \end{bmatrix} \tag{3}$$

The shape functions of the axisymmetric triangle are defined as:

$$N_i = \frac{1}{2A^e}(a_i + b_i r + c_i z) \tag{4}$$

where:
$a_i = r_j z_k - r_k z_j$
$b_i = r_j z_k - r_k z_j$
$b_i = r_j z_k - r_k z_j$

In our case:

Nodes	r	z	a_i	b_i	c_i
1	0	0	ab	-b	0
2	a	0	0	b	-a
3	a	b	0	0	a

Thus,

$$N_1 = \frac{1}{2A}(ab - br) = 1 - \frac{r}{a} \tag{5}$$

$$N_2 = \frac{1}{2A}(br - az) = \frac{r}{a} - \frac{z}{b} \tag{6}$$

$$N_3 = \frac{1}{2A}(az) = \frac{z}{b} \tag{7}$$

Taking into account that $A = \dfrac{ab}{2}$ the matrix B is:

$$B_i = \left[\begin{array}{cc|cc|cc} \dfrac{-1}{a} & 0 & \dfrac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{-1}{b} & 0 & \dfrac{1}{b} \\ \dfrac{a-r}{ar} & 0 & \dfrac{br-az}{abr} & 0 & \dfrac{z}{br} & 0 \\ 0 & \dfrac{-1}{a} & \dfrac{-1}{b} & \dfrac{1}{a} & \dfrac{1}{b} & 0 \end{array}\right] \tag{8}$$

Thus, the stiffness matrix is:

$$K^e = \int_A 2\pi E \begin{bmatrix} \dfrac{-2}{a} + \dfrac{2r}{a^2} + \dfrac{1}{r} & 0 & \dfrac{z}{ab} + \dfrac{1}{a} - \dfrac{2r}{a^2} - \dfrac{z}{br} & 0 & \dfrac{z}{br} - \dfrac{z}{ab} & 0 \\ 0 & \dfrac{r}{2a^2} & \dfrac{r}{2ab} & -\dfrac{r}{2a^2} & -\dfrac{r}{2ab} & 0 \\ \dfrac{z}{ab} + \dfrac{1}{a} - \dfrac{2r}{a^2} - \dfrac{z}{br} & \dfrac{r}{2ab} & -\dfrac{2z}{ab} + \dfrac{2r}{a^2} + \dfrac{r}{2b^2} + \dfrac{z^2}{rb^2} & -\dfrac{r}{2ab} & \dfrac{z}{ab} - \dfrac{r}{2b^2} - \dfrac{z^2}{rb^2} & 0 \\ 0 & -\dfrac{r}{2a^2} & -\dfrac{r}{2ab} & \dfrac{r}{2a^2} + \dfrac{r}{b^2} & \dfrac{r}{2ab} & -\dfrac{r}{b^2} \\ \dfrac{z}{br} - \dfrac{z}{ab} & -\dfrac{r}{2ab} & \dfrac{z}{ab} - \dfrac{r}{2b^2} - \dfrac{z^2}{rb^2} & \dfrac{r}{2ab} & \dfrac{r}{2b^2} + \dfrac{z^2}{rb^2} & 0 \\ 0 & 0 & 0 & -\dfrac{r}{b^2} & 0 & \dfrac{r}{b^2} \end{bmatrix} dA$$

To integrate the stiffness matrix, numerical integration will be performed by means of Gauss quadratures. First, it is necessary to transform the area integral into two integrals with normalized limits.

The following transformation is necessary:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \\ 1 - \xi - \eta \end{bmatrix} \tag{9}$$

The linear approximation is:

$$r = N_1 r_1 + N_2 r_2 + N_3 r_3 \tag{10}$$

$$z = N_1 z_1 + N_2 z_2 + N_3 z_3 \tag{11}$$

Thus,

$$r = (r_1 - r_3)\xi + (r_2 - r_3)_\eta + r_3 \tag{12}$$

$$z = (z_1 - z_3)\xi + (z_2 - z_3)_\eta + z_3 \tag{13}$$

The Jacobian is:

$$\begin{bmatrix} \dfrac{\delta r}{\delta \xi} & \dfrac{\delta z}{\delta \xi} \\ \dfrac{\delta r}{\delta \eta} & \dfrac{\delta z}{\delta \eta} \end{bmatrix} = \begin{bmatrix} -a & -b \\ 0 & -b \end{bmatrix} \tag{14}$$

It is used one Gauss' point located in $r = \dfrac{2a}{3}$ and $z = \dfrac{b}{3}$.

Finally, neglecting the 2π parameter:

$$K^e = E \begin{bmatrix} \frac{5b}{6} & 0 & -\frac{b}{2} & 0 & \frac{b}{6} & 0 \\ 0 & \frac{b}{3} & \frac{a}{3} & -\frac{b}{3} & -\frac{a}{3} & 0 \\ -\frac{b}{2} & \frac{a}{3} & \frac{a^2}{3b} + \frac{5b}{6} & -\frac{a}{3} & \frac{b}{6} - \frac{a^2}{3b} & 0 \\ 0 & -\frac{b}{3} & -\frac{a}{3} & \frac{2a^2}{3b} + \frac{b}{3} & \frac{a}{3} & -\frac{2a^2}{3b} \\ \frac{b}{6} & -\frac{a}{3} & \frac{b}{6} - \frac{a^2}{3b} & \frac{a}{3} & \frac{a^2}{3b} + \frac{b}{6} & 0 \\ 0 & 0 & 0 & -\frac{2a^2}{3b} & 0 & \frac{2a^2}{3b} \end{bmatrix} \tag{15}$$

4.1.2 Show that the sum of the rows (and columns) 2, 4 and 6 of K^e must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.

The sum of the rows (and columns) 2, 4 and 6 is equal to zero.

This nodes are free to move and due to this the rigidity of those degrees of freedom are equal to zero, allowing to have motion.

The sum of rows (and columns) 1, 3 and 5 is equal to b.

The nodes are not allowed to move completely free through this direction due to the rigidity imposed.

4.1.3 Compute the consistent force vector f^e for gravity forces $b = [0, g]^T$.

The force vector is defined by the following equation:

$$f_b = \int_A rNbdA \tag{16}$$

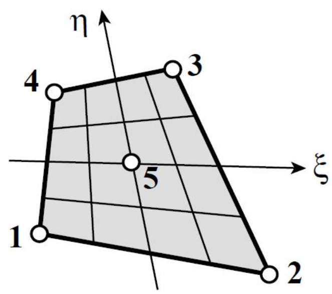
Now, substituting the shape functions:

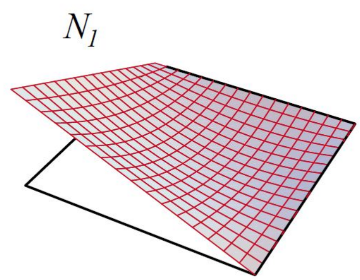
$$f_b = \int_A -g \begin{bmatrix} 0 \\ r - \dfrac{r^2}{a} \\ 0 \\ \dfrac{r^2}{a} - \dfrac{zr}{b} \\ 0 \\ \dfrac{zr}{b} \end{bmatrix} dA \tag{17}$$

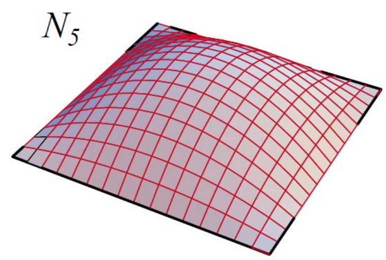
Integrating as in the stiffness matrix:

$$f_b = -g \begin{bmatrix} 0 \\ \dfrac{a^2b}{9} \\ 0 \\ \dfrac{a^2b}{9} \\ 0 \\ \dfrac{a^2b}{9} \end{bmatrix} dA \tag{18}$$

4.2 A five node quadrilateral element has the nodal configuration shown in the figure. Perspective views of N_1^e and N_5^e are shown in the same figure. Find five shape functions N_i^e, i=1,,5 that satisfy compatibility and also verify that their sum is unity.







$$N_5(\xi, \eta) = C_5\, L_{1-2}\, L_{2-3}\, L_{3-4}\, L_{4-1}$$

For sides 1-2, 2-3, 3-4 and 4-1 the values can be $\xi = -1$ or $\xi = 1$, $\eta = -1$ or $\eta = 1$, $\xi = 1$ or $\xi = -1$ and $\eta = 1$ or $\eta = -1$, respectively. Replacing in the previous expression:

$$N_5(\xi, \eta) = c_5(1 + \eta)(1 - \xi)(1 - \eta)(1 + \xi)$$

$$N_5|_{\text{at node 5}} = 1 = c_5(1 + 0)(1 - 0)(1 - 0)(1 + 0) = 1 \rightarrow c_5 = 1$$

Thus,

$$\boxed{N_5(\xi, \eta) = (1 - \xi^2)(1 - \eta^2)} \tag{19}$$

The remaining shape functions will be:

$$N_i = \overline{N_i} + \alpha N_5$$

For node 1:

$$N_1 = \overline{N_1} + \alpha N_5 = \frac{1}{4}(1 - \xi)(1 - \eta) + \alpha(1 - \xi^2)(1 - \eta^2)$$

$\eta = \xi = 0$ at node 5:

$$\frac{1}{4}(1 - 0)(1 - 0) + \alpha(1 - 0^2)(1 - 0^2) = 0 \rightarrow \boxed{\alpha = -\frac{1}{4}}$$

which will be the same for all N_i. Thus,

$$\boxed{N_1 = \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2)} \tag{20}$$

For node 2:

$$\begin{aligned} N_2 &= N_2 - \frac{1}{4}N_5 \\ &= \frac{1}{4}(1+\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2) \\ &= \frac{1}{4}(1+\xi)(1-\eta)[1-(1-\xi)(1+\eta)] \end{aligned}$$

$$\boxed{N_2 = \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta+\xi\eta)} \tag{21}$$

For node 3:

$$\begin{aligned} N_3 &= N_3 - \frac{1}{4}N_5 \\ &= \frac{1}{4}(1+\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2) \\ &= \frac{1}{4}(1+\xi)(1+\eta)[1-(1-\xi)(1-\eta)] \end{aligned}$$

$$\boxed{N_3 = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-\xi\eta)} \tag{22}$$

For node 4:

$$\begin{aligned} N_4 &= N_4 - \frac{1}{4}N_5 \\ &= \frac{1}{4}(1-\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2) \\ &= \frac{1}{4}(1-\xi)(1+\eta)[1-(1+\xi)(1-\eta)] \end{aligned}$$

$$\boxed{N_4 = \frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta+\xi\eta)} \tag{23}$$

These four corner shape functions vary linearly over all sides because the bubble function N_5 vanishes over the 4 sides. Hence interelement continuity is maintained. Unit sum check: $N_1 + N_2 + N_3 + N_4 + N_5 = \overline{N_1} + \overline{N_2} + \overline{N_3} + \overline{N_4} - 4 \cdot 1/4\, N_5 + N_5 = \overline{N_1} + \overline{N_2} + \overline{N_3} + \overline{N_4} = 1$, since it is known that the shape functions $\overline{N_i}$ of the 4-node quad add up to one.