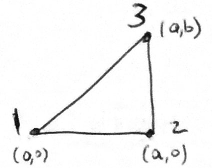
Assignment 4.1



$$A^e = \frac{ab}{2}$$

$$\begin{bmatrix} u_r \\ u_z \end{bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$D = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \qquad N = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix}$$

$$B = DN = \begin{bmatrix} \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial r} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \frac{\partial N_3}{\partial z} \\ \frac{N_1}{r} & \frac{N_2}{r} & \frac{N_3}{r} & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \frac{\partial N_3}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial r} \end{bmatrix}$$

$$N_i = \frac{1}{2A}(\alpha_i + r\beta_i + z\gamma_i)$$

$$\alpha_i = r_j z_k - r_k z_j$$

$$\beta_i = z_j - z_k$$

$$\gamma_i = r_k - r_j$$

$$B = \frac{1}{2A^e} \begin{bmatrix} -b & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a & a \\ \frac{ab-br}{r} & \frac{br-az}{r} & \frac{az}{r} & 0 & 0 & 0 \\ 0 & -a & a & -b & b & 0 \end{bmatrix} \qquad E = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$$B^T E B = \frac{E}{4A^2} \begin{bmatrix} -b & 0 & \frac{ab-br}{r} & 0 \\ b & 0 & \frac{br-az}{r} & -a \\ 0 & 0 & \frac{az}{r} & a \\ 0 & 0 & 0 & -b \\ 0 & -a & 0 & b \\ 0 & a & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} -b & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a & a \\ \frac{ab-br}{r} & \frac{br-az}{r} & \frac{az}{r} & 0 & 0 & 0 \\ 0 & -a & a & -b & b & 0 \end{bmatrix}$$

$$= \frac{2\pi E}{a^2 b^2} \int_0^b \int_0^a \begin{bmatrix} \frac{(ab-br)^2}{r^2} + b^2 & \frac{(ab-br)(br-az)}{r^2} - b^2 & \frac{a^2bz - abrz}{r^2} & 0 & 0 & 0 \\ \frac{(ab-br)(br-az)}{r^2} - b^2 & \frac{a^2}{2} + b^2 + \frac{(br-az)^2}{r^2} & \frac{abrz - a^2z^2}{r^2} - \frac{a^2}{2} & \frac{ab}{2} & -\frac{ab}{2} & 0 \\ \frac{a^2bz - abrz}{r^2} & \frac{abrz - a^2z^2}{r^2} - \frac{a^2}{2} & \frac{a^2z^2}{r^2} + \frac{a^2}{2} & -\frac{ab}{2} & \frac{ab}{2} & 0 \\ 0 & \frac{ab}{2} & -\frac{ab}{2} & \frac{b^2}{2} & -\frac{b^2}{2} & 0 \\ 0 & -\frac{ab}{2} & \frac{ab}{2} & -\frac{b^2}{2} & a^2 + \frac{b^2}{2} & -a^2 \\ 0 & 0 & 0 & 0 & -a^2 & a^2 \end{bmatrix} r\,dr\,dz$$

$$k^e = \underline{B}^T \underline{D}\, \underline{B}\; 2\pi \bar{r} A \qquad \underline{B} \text{ matrix at centroid} \qquad \bar{r} = \frac{2a}{3}$$

$$k^e = \pi E b \begin{bmatrix} 5/6 & 2/3 & 1/3 & 0 & 0 & 0 \\ 2/3 & \frac{a^2}{3b^2}+\frac{2}{3} & -\frac{a^2}{2b^2} & \frac{a}{2b} & -\frac{a}{2b} & 0 \\ 1/3 & -\frac{a^2}{2b^2} & \frac{a^2}{2b^2}+1 & -\frac{a}{2b} & \frac{a}{2b} & 0 \\ 0 & \frac{a}{2b} & -a/2b & 1/2 & -1/2 & 0 \\ 0 & -\frac{a}{2b} & a/2b & -1/2 & \frac{a^2}{b^2}+1/2 & -a^2/b^2 \\ 0 & 0 & 0 & 0 & -\frac{a^2}{b^2} & a^2/b^2 \end{bmatrix}$$

$$f^e_{ext} = \int_A N^T b\, r\, dA$$

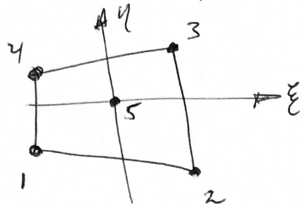
$$N^T b = \begin{bmatrix} N_1 & 0 \\ N_2 & 0 \\ N_3 & 0 \\ 0 & N_1 \\ 0 & N_2 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} 0 \\ -g \end{bmatrix} = -g \begin{bmatrix} 0 \\ 0 \\ 0 \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}$$

$$f^e_{ext} = -g \int_0^{br/a}\int_0^a \begin{bmatrix} 0 \\ 0 \\ 0 \\ ab-br \\ br-az \\ az \end{bmatrix} dr\, dz = -g\int_0^a \begin{bmatrix} 0 \\ 0 \\ 0 \\ abz-brz \\ brz-\frac{1}{2}az^2 \\ \frac{1}{2}az^2 \end{bmatrix} dr = -g\int_0^a \begin{bmatrix} 0 \\ 0 \\ 0 \\ b^2r-\frac{b^2r^2}{a} \\ \frac{b^2r^2}{a}-\frac{1}{2}\frac{b^2r^2}{a} \\ \frac{1}{2}\frac{b^2r^2}{a} \end{bmatrix} dr$$

$$f^e_{ext} = -g \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{b^2r^2}{2}-\frac{b^2r^3}{3a} \\ \frac{b^2r^3}{3a}-\frac{b^2r^3}{6a} \\ \frac{b^2r^3}{6a} \end{bmatrix} = -g \begin{bmatrix} 0 \\ 0 \\ 0 \\ a^2b^2/6 \\ a^2b^2/6 \\ a^2b^2/6 \end{bmatrix}$$

Assignment 4.2

Find five shape functions N_i^e, $i=1,\dots,5$ that satisfy compatibility and also verify that their sum is unity



$$N_5^e = (1-\xi)(1-\eta)(1+\xi)(1+\eta)$$
$$N_5^e = (1-\xi^2)(1-\eta^2)$$

$$N_i = \bar{N}_i + \alpha N_5$$
$$\xi = \eta = 0 \Rightarrow N_i = 0$$
$$0 = \frac{1}{4} + \alpha$$
$$\alpha = -\frac{1}{4}$$

$$\boxed{\begin{aligned}
N_1^e &= \tfrac{1}{4}(1-\xi)(1-\eta) - \tfrac{1}{4}(1-\xi^2)(1-\eta^2)\\
N_2^e &= \tfrac{1}{4}(1+\xi)(1-\eta) - \tfrac{1}{4}(1-\xi^2)(1-\eta^2)\\
N_3^e &= \tfrac{1}{4}(1+\xi)(1+\eta) - \tfrac{1}{4}(1-\xi^2)(1-\eta^2)\\
N_4^e &= \tfrac{1}{4}(1-\xi)(1+\eta) - \tfrac{1}{4}(1-\xi^2)(1-\eta^2)\\
N_5^e &= (1-\xi^2)(1-\eta^2)
\end{aligned}}$$

	$\xi=-1$, $\eta=-1$	$\xi=1$, $\eta=-1$	$\xi=1$, $\eta=1$	$\xi=-1$, $\eta=1$	$\xi=0$, $\eta=0$
N_1	1	0	0	0	0
N_2	0	1	0	0	0
N_3	0	0	1	0	0
N_4	0	0	0	1	0
N_5	0	0	0	0	1

$$\begin{aligned}
\Sigma N_i = &\tfrac{1}{4}(1-\xi-\eta+\xi\eta) - \tfrac{1}{4}(1-\xi^2)(1-\eta^2)\\
&+ \tfrac{1}{4}(1+\xi-\eta-\xi\eta) - \tfrac{1}{4}(1-\xi^2)(1-\eta^2)\\
&+ \tfrac{1}{4}(1+\xi+\eta+\xi\eta) - \tfrac{1}{4}(1-\xi^2)(1-\eta^2)\\
&+ \tfrac{1}{4}(1-\xi+\eta-\xi\eta) - \tfrac{1}{4}(1-\xi^2)(1-\eta^2)\\
&+ (1-\xi^2)(1-\eta^2)
\end{aligned}$$

$$\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4} = 1$$

$$\boxed{\sum_1^5 N_i = 1}$$