Universitat Politècnica De Catalunya, Barcelona

MSc. Computational Mechanics Erasmus Mundus

Assignment 4: Structures of revolution

Computational Structural Mechanics & Dynamics

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Date: March 4, 2018

Assignment 4.1

On "Structures of revolution":

1. Compute the entries of $\boldsymbol{K}^e$ for the following axisymmetric triangle:

$$r_1 = 0, \quad r_2 = r_3 = a, \quad z_1 = z_2 = 0, \quad z_3 = b$$

The material is isotropic with $\nu = 0$ for which the stress-strain matrix is,

$$\boldsymbol{E} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Solution: The stiffness matrix for an axisymmetric element is given by,

$$\boldsymbol{K}^e = \int_{\Omega^e} \boldsymbol{B}^{e\,T} \boldsymbol{E} \boldsymbol{B}^e r \, d\Omega \tag{1}$$

where, the factor 2π is neglected for this problem and the $\boldsymbol{B}^e$ matrix is given as,

$$\boldsymbol{B}^e = \boldsymbol{D}\boldsymbol{N} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

We get,

$$\boldsymbol{B}^e = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} \end{bmatrix} \tag{2}$$

Therefore, to evaluate $\boldsymbol{B}^e$ in the above equation (2), first we need the shape functions and their derivatives in the global coordinates r, z although they are defined in the reference/local coordinates ξ, η. The shape functions for an iso-parametric triangle are given as,

$$N_1 = \xi, \quad N_2 = \eta, \quad N_3 = 1 - \xi - \eta$$

Using the properties of shape functions,

$$r = \sum_{i}^{3} r_i N_i, \quad z = \sum_{i}^{3} z_i N_i \quad \text{and} \quad N_1 + N_2 + N_3 = 1$$

we find the shape functions in global coordinates as,

$$N_1 = 1 - \frac{r}{a}, \quad N_2 = \frac{r}{a} - \frac{z}{b} \quad \text{and} \quad N_3 = \frac{z}{b} \tag{3}$$

Now, the derivatives of the shape functions in reference coordinates are given as,

$$\frac{\partial N_1}{\partial \xi} = 1, \quad \frac{\partial N_2}{\partial \xi} = 0, \quad \frac{\partial N_3}{\partial \xi} = -1$$

$$\frac{\partial N_1}{\partial \eta} = 0, \quad \frac{\partial N_2}{\partial \eta} = 1, \quad \frac{\partial N_3}{\partial \eta} = -1$$

So, we use the Jacobian matrix for the transformation of coordinate system,

$$\boldsymbol{J} = \begin{bmatrix} \dfrac{\partial r}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial r}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \displaystyle\sum_{i}^{3} r_i \frac{\partial N_i}{\partial \xi} & \displaystyle\sum_{i}^{3} z_i \frac{\partial N_i}{\partial \xi} \\ \displaystyle\sum_{i}^{3} r_i \frac{\partial N_i}{\partial \eta} & \displaystyle\sum_{i}^{3} z_i \frac{\partial N_i}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -a & -b \\ 0 & -b \end{bmatrix}$$

The derivatives of the shape functions with respect to the global coordinates are computed using the inverse of Jacobian matrix as,

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial r} \\ \dfrac{\partial N_i}{\partial z} \end{bmatrix} = \boldsymbol{J}^{-1} \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{bmatrix}$$

where, the inverse is calculated as,

$$\boldsymbol{J}^{-1} = \begin{bmatrix} \dfrac{-1}{a} & \dfrac{1}{a} \\ 0 & \dfrac{-1}{b} \end{bmatrix}$$

Thus, for N_1, N_2 and N_3, we get the derivatives as,

$$\begin{bmatrix} \dfrac{\partial N_1}{\partial r} \\ \dfrac{\partial N_1}{\partial z} \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{a} & \dfrac{1}{a} \\ 0 & \dfrac{-1}{b} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{a} \\ 0 \end{bmatrix} \tag{4.1}$$

$$\begin{bmatrix} \dfrac{\partial N_2}{\partial r} \\ \dfrac{\partial N_2}{\partial z} \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{a} & \dfrac{1}{a} \\ 0 & \dfrac{-1}{b} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{a} \\ \dfrac{-1}{b} \end{bmatrix} \tag{4.2}$$

$$\begin{bmatrix} \dfrac{\partial N_3}{\partial r} \\ \dfrac{\partial N_3}{\partial z} \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{a} & \dfrac{1}{a} \\ 0 & \dfrac{-1}{b} \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{1}{b} \end{bmatrix} \tag{4.3}$$

Finally, using the values derived in equations (3),(4.1), (4.2) and (4.3), we evaluate the $\boldsymbol{B}^e$ matrix as,

$$\boldsymbol{B}^e = \begin{bmatrix} \dfrac{-1}{a} & 0 & \dfrac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{-1}{b} & 0 & \dfrac{1}{b} \\ \left(\dfrac{1}{r} - \dfrac{1}{a}\right) & 0 & \left(\dfrac{1}{a} - \dfrac{z}{rb}\right) & 0 & \dfrac{z}{rb} & 0 \\ 0 & \dfrac{-1}{a} & \dfrac{-1}{b} & \dfrac{1}{a} & \dfrac{1}{b} & 0 \end{bmatrix} \tag{5}$$

Next, before computing the integral to find $\boldsymbol{K}^e$, we find,

$$r\boldsymbol{B}^{eT}\boldsymbol{E}\boldsymbol{B}^e = r \begin{bmatrix} \dfrac{-1}{a} & 0 & \left(\dfrac{1}{r} - \dfrac{1}{a}\right) & 0 \\ 0 & 0 & 0 & \dfrac{-1}{a} \\ \dfrac{1}{a} & 0 & \left(\dfrac{1}{a} - \dfrac{z}{rb}\right) & \dfrac{-1}{b} \\ 0 & \dfrac{-1}{b} & 0 & \dfrac{1}{a} \\ 0 & 0 & \dfrac{z}{rb} & \dfrac{1}{b} \\ 0 & \dfrac{1}{b} & 0 & 0 \end{bmatrix} E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dfrac{1}{2} \end{bmatrix} \begin{bmatrix} \dfrac{-1}{a} & 0 & \dfrac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{-1}{b} & 0 & \dfrac{1}{b} \\ \left(\dfrac{1}{r} - \dfrac{1}{a}\right) & 0 & \left(\dfrac{1}{a} - \dfrac{z}{rb}\right) & 0 & \dfrac{z}{rb} & 0 \\ 0 & \dfrac{-1}{a} & \dfrac{-1}{b} & \dfrac{1}{a} & \dfrac{1}{b} & 0 \end{bmatrix}$$

$$= E \begin{bmatrix} \left(\frac{2r}{a^2}+\frac{1}{r}-\frac{2}{a}\right) & 0 & \left(\frac{-2r}{a^2}+\frac{1}{a}-\frac{z}{rb}+\frac{z}{ab}\right) & 0 & \left(\frac{z}{rb}-\frac{z}{ab}\right) & 0 \\ 0 & \frac{r}{2a^2} & \frac{r}{2ab} & \frac{-r}{2a^2} & \frac{-r}{2ab} & 0 \\ \left(\frac{-2r}{a^2}+\frac{1}{a}-\frac{z}{rb}+\frac{z}{ab}\right) & \frac{r}{2ab} & \left(\frac{2r}{a^2}+\frac{r}{2b^2}+\frac{z^2}{rb^2}-\frac{2z}{ab}\right) & \frac{-r}{2ab} & \left(\frac{z}{ab}-\frac{z^2}{rb^2}-\frac{r}{2b^2}\right) & 0 \\ 0 & \frac{-r}{2a^2} & \frac{-r}{2ab} & \left(\frac{r}{2a^2}+\frac{r}{b^2}\right) & \frac{r}{2ab} & \frac{-r}{b^2} \\ (\frac{z}{rb}-\frac{z}{ab} & \frac{-r}{2ab}) & \left(\frac{z}{ab}-\frac{z^2}{rb^2}-\frac{r}{2b^2}\right) & \frac{r}{2ab} & \left(\frac{z^2}{rb^2}+\frac{r}{2b^2}\right) & 0 \\ 0 & 0 & 0 & \frac{-r}{b^2} & 0 & \frac{r}{b^2} \end{bmatrix} \tag{6}$$

Now, we solve the integral as in the equation (1) for evaluation of elemental stiffness matrix. It is interesting to note that integral of a few terms in the above matrix is undefined since natural logarithm function $\ln(x)$ is defined only for $x > 0$. Therefore, for the integral terms containing $1/r$, we will have to use one of the Gaussian quadrature rules. In this case we use the centroid rule (1 point, degree 1) to get an approximate solution which is given as,

$$\frac{1}{A}\int_{\Omega^e} F\left(N_1, N_2, N_3\right) d\Omega \approx F\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

where $A = \dfrac{ab}{2}$

In order to implement this, we use the relations shown in equation (3) to substitute $r = a(N_2 + N_3)$ and $z = bN_3$ in each term of the matrix shown above in equation (6).

Finally, the derived elemental stiffness matrix from

$$\boldsymbol{K}^e = \frac{ab}{2} F\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

is given as,

$$\boldsymbol{K}^e = \frac{Eab}{2} \begin{bmatrix} \frac{5}{6a} & 0 & \frac{-1}{2a} & 0 & \frac{1}{6a} & 0 \\ 0 & \frac{1}{3a} & \frac{1}{3b} & \frac{-1}{3a} & \frac{-1}{3b} & 0 \\ \frac{-1}{2a} & \frac{1}{3b} & \left(\frac{5}{6a}+\frac{a}{3b^2}\right) & \frac{-1}{3b} & \left(\frac{1}{6a}-\frac{a}{3b^2}\right) & 0 \\ 0 & \frac{-1}{3a} & \frac{-1}{3b} & \left(\frac{2a}{3b^2}+\frac{1}{3a}\right) & \frac{1}{3b} & \frac{-2a}{3b^2} \\ \frac{1}{6a} & \frac{-1}{3b} & \left(\frac{1}{6a}-\frac{a}{3b^2}\right) & \frac{1}{3b} & \left(\frac{1}{6a}+\frac{a}{3b^2}\right) & 0 \\ 0 & 0 & 0 & \frac{-2a}{3b^2} & 0 & \frac{2a}{3b^2} \end{bmatrix} \tag{7}$$

2. Show that the sum of the rows (and columns) 2, 4 and 6 of $\boldsymbol{K}^e$ must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.

Solution: Considering the terms of $\boldsymbol{K}^e$ obtained in equation (7), we find the sum of rows (and columns) 2, 4 and 6 and it is clearly observed that,

$$\sum_{j=1}^{6} k_{2j} = \sum_{j=1}^{6} k_{4j} = \sum_{j=1}^{6} k_{6j} = 0$$

$$\sum_{i=1}^{6} k_{i2} = \sum_{i=1}^{6} k_{i4} = \sum_{i=1}^{6} k_{i6} = 0$$

On the other hand, the sum of rows (and columns) 1, 3 and 5 is not equal to zero, i.e.

$$\sum_{j=1}^{6} k_{1j} \neq 0, \quad \sum_{j=1}^{6} k_{3j} \neq 0, \quad \sum_{j=1}^{6} k_{5j} \neq 0$$

$$\sum_{i=1}^{6} k_{i1} \neq 0, \quad \sum_{i=1}^{6} k_{i3} \neq 0, \quad \sum_{i=1}^{6} k_{i5} \neq 0$$

It is important to note that we chose to define the displacement vector $\boldsymbol{u}^e$ as,

$$\boldsymbol{u}^e = \begin{bmatrix} u_{r1} \\ u_{z1} \\ u_{r2} \\ u_{z2} \\ u_{r3} \\ u_{z3} \end{bmatrix}$$

Therefore, the rows (and columns) 1, 3 and 5 correspond to the radial displacement components and rows (and columns) 2, 4 and 6 correspond to the vertical displacement components. The sum of rows (and columns) of the stiffness matrix correlate to the ability of generating rigid body motion in that direction, but due to the concept of hoop strain in structures of revolution, the sum of rows (and columns) related to radial displacement i.e. 1, 3 and 5 does not disappear whereas the possibility of generating rigid body motion in the vertical displacement direction makes the sum of row (and column) 2, 4 and 6 disappear in the stiffness matrix as shown above.

3. Compute the consistent force vector $\boldsymbol{f}^e$ for gravity forces $\boldsymbol{b} = [0, -g]^T$.

<u>**Solution**</u>**:** The consistent nodal force vector for body load is given as,

$$\boldsymbol{f}^e = \int_{\Omega^e} \boldsymbol{N}^T \boldsymbol{b} r d\Omega \tag{8}$$

where the factor 2π is neglected with $\boldsymbol{b}$ and $\boldsymbol{N}^T$ given as,

$$\boldsymbol{b} = \begin{bmatrix} 0 \\ -g \end{bmatrix}, \quad \boldsymbol{N}^T = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix}$$

Also, from equation (3), we know that,

$$r = a\,(N_2 + N_3)$$

Using these in equation (8), we get,

$$\boldsymbol{f}^e = \int_{\Omega^e} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} 0 \\ -g \end{bmatrix} a(N_2 + N_3)\, d\Omega = \int_{\Omega^e} \begin{bmatrix} 0 \\ -g\ N_1\ a(N_2 + N_3) \\ 0 \\ -g\ N_2\ a(N_2 + N_3) \\ 0 \\ -g\ N_3\ a(N_2 + N_3) \end{bmatrix} d\Omega$$

Noting that the degree of polynomial in the integrand is 2, in this case we use the midpoint Gaussian quadrature rule (3 points degree 2) given as,

$$\frac{1}{A} \int_{\Omega^e} F\ (N_1, N_2, N_3)\ d\Omega \approx \frac{1}{3}\ F\left(\frac{1}{2}, \frac{1}{2}, 0\right) + \frac{1}{3}\ F\left(0, \frac{1}{2}, \frac{1}{2}\right) + \frac{1}{3}\ F\left(\frac{1}{2}, 0, \frac{1}{2}\right)$$

Hence, the consistent force vector $\boldsymbol{f}^e$ is given as,

$$\boldsymbol{f}^e = \begin{bmatrix} 0 \\ \dfrac{-a^2 bg}{12} \\ 0 \\ \dfrac{-a^2 bg}{8} \\ 0 \\ \dfrac{-a^2 bg}{8} \end{bmatrix} \tag{9}$$