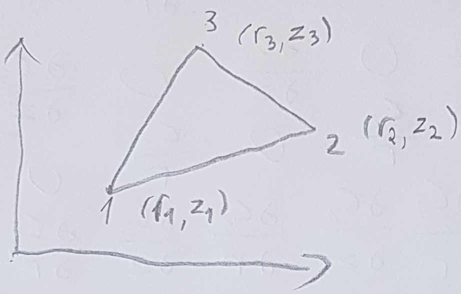
Assignment 4

$r_1 = 0 \quad r_2 = r_3 = a \quad z_1 = z_2 = 0 \quad z_3 = b$

1) For $\nu = 0$

$$E = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}, \quad \begin{Bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ \gamma_{rz} \end{Bmatrix} = e = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} u_r \\ u_z \end{bmatrix}$$



$$\begin{bmatrix} u_r \\ u_z \end{bmatrix} = \begin{bmatrix} u_{r1} \\ u_{z1} \\ u_{r2} \\ u_{z2} \\ u_{r3} \\ u_{z3} \end{bmatrix}$$

a) $A = \frac{1}{2} \det \begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$

$$A = \frac{1}{2}(a \times b) = \frac{ab}{2}$$

$$[N] = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} r_2 z_3 - r_3 z_2 & z_{23} & r_{32} \\ r_3 z_1 - r_1 z_3 & z_{31} & r_{13} \\ r_1 z_2 - r_2 z_1 & z_{12} & r_{21} \end{bmatrix} \begin{bmatrix} 1 \\ r \\ z \end{bmatrix}$$

$z_{ij} = z_i - z_j \qquad \xi_1 + \xi_2 + \xi_3 = 1$

$r_{ij} = r_i - r_j \qquad$ (natural coordinates)

$$[N] = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{a} & 0 \\ 0 & \frac{1}{a} & -1/b \\ 0 & 0 & 1/b \end{bmatrix} \begin{bmatrix} 1 \\ r \\ z \end{bmatrix} \Rightarrow$$

natural coordinates

$$\Rightarrow \begin{pmatrix} 1 - \frac{r}{a} \\ \frac{r}{a} - \frac{z}{b} \\ \frac{z}{b} \end{pmatrix}$$

$$[B] = \begin{bmatrix} \frac{\partial \zeta_1}{\partial r} & 0 & \frac{\partial \zeta_2}{\partial r} & 0 & \frac{\partial \zeta_3}{\partial r} & 0 \\ 0 & \frac{\partial \zeta_1}{\partial z} & 0 & \frac{\partial \zeta_2}{\partial z} & 0 & \frac{\partial \zeta_3}{\partial z} \\ \frac{\zeta_1}{r} & 0 & \frac{\zeta_2}{r} & 0 & \frac{\zeta_3}{r} & 0 \\ \frac{\partial \zeta_1}{\partial z} & \frac{\partial \zeta_1}{\partial r} & \frac{\partial \zeta_2}{\partial z} & \frac{\partial \zeta_2}{\partial r} & \frac{\partial \zeta_3}{\partial z} & \frac{\partial \zeta_3}{\partial r} \end{bmatrix}$$

strain displacement Matrix

$$B = \begin{bmatrix} -\frac{1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b} & 0 & \frac{1}{b} \\ -\frac{\frac{r}{a}-1}{r} & 0 & \frac{\frac{r}{a}-\frac{z}{b}}{r} & 0 & \frac{z}{b-r} & 0 \\ 0 & -\frac{1}{a} & -\frac{1}{b} & \frac{1}{a} & \frac{1}{b} & 0 \end{bmatrix}$$

$$K^e = \int_{\Omega^e} r\,(B)^T\, \underline{E}\, (\underline{B})\, d\Omega$$

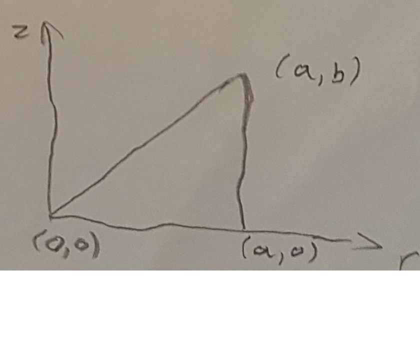
$$K^e = \sum_{k=1}^{?} w_k\, B^T(\xi_k)\, r(\xi_k)\, J_\Omega\, (\xi_k)$$

$$r(\zeta_i) = \sum_{i=1}^{n} r_i N_i^{(e)} (\zeta_i) = \qquad r = r_1 \zeta_1 + r_2 \zeta_2 + r_3 \zeta_3$$

$$z = z_1 \zeta_1 + z_2 \zeta_2 + z_3 \zeta_3$$

$$d\Omega = J\, d\zeta_1\, d\zeta_2\, d\zeta_3$$

$$J = A$$



For gauss points $\zeta_1 = \zeta_2 = \zeta_3 = \frac{1}{3}$

$$\bar{r} = \frac{r_1 + r_2 + r_3}{3} = \frac{2a}{3}$$

$$\bar{z} = \frac{z_1 + z_2 + z_3}{3} = \frac{b}{3}$$

$$\Delta = J = A$$

$$K^e = B^T D B \overset{2a/3}{\bar{r}} \overset{A}{\Delta} \Rightarrow$$

$$K^e = \begin{bmatrix} \frac{5Eb}{12} & 0 & -\frac{Eb}{4} & 0 & \frac{Eb}{12} & 0 \\ 0 & \frac{Eb}{6} & \frac{Ea}{6} & -\frac{Eb}{6} & -\frac{Ea}{6} & 0 \\ -\frac{Eb}{4} & \frac{Ea}{6} & \frac{E(2a^2+5b)}{12b} & -\frac{Ea}{6} & \frac{E(b^2-2a^2)}{12b} & 0 \\ 0 & -\frac{Eb}{6} & -\frac{Ea}{6} & \frac{E(2a^2+b^2)}{6b} & \frac{Ea}{6} & -\frac{Ea^2}{3b} \\ \frac{Eb}{12} & -\frac{Ea}{6} & \frac{E(b^2-2a^2)}{12b} & \frac{Ea}{6} & \frac{E(2a^2+b^2)}{12b} & 0 \\ 0 & 0 & 0 & -\frac{Ea^2}{3b} & 0 & \frac{Ea^2}{3b} \end{bmatrix}$$

b). Sum of row 2

$$\frac{Eb}{6} + \frac{Ea}{6} - \frac{Eb}{6} - \frac{Ea}{6} = 0$$

Sum row 4

$$-\frac{Eb}{6} - \frac{Ea}{6} + \frac{E(2a^2+b^2)}{6b} + \frac{Ea}{6} - \frac{Ea^2}{3b} = 0$$

Sum of row 6

$$-\frac{Ea^2}{3b} + \frac{Ea^2}{3b} = 0$$

It allows rigid body motion in 3-direction

Sum of row ~~#~~1

$$\frac{5Eb}{12} - \frac{Eb}{4} + \frac{Eb}{12} = \frac{Eb}{4}$$

Sum of row 3

$$-\frac{Eb}{4} + \frac{Ea}{6} + \frac{E(2a^2+5b)}{12b} - \frac{Ea}{6} + \frac{E(b^2-2a^2)}{12b} = \frac{Eb}{4}$$

Sum of row 5

$$\frac{Eb}{12} - \frac{Ea}{6} + \frac{E(b^2-2a^2)}{12b} + \frac{Ea}{6} + \frac{E(2a^2+b^2)}{12b} = \frac{Eb}{4}$$

It's not possible to make a rigid body movement in r-axis.

c) Body forces

$$f^e = \int I N_i \begin{Bmatrix} b_r \\ b_z \end{Bmatrix} r \, dr \, dz$$

$$N_i = \frac{a_i + b_i r + c_i z}{3}$$

using $r = \bar{r} = \dfrac{r_1 + r_2 + r_3}{3}$

$$f^e = \begin{Bmatrix} b_r \\ b_z \end{Bmatrix} \frac{\bar{r} A}{3} = \begin{Bmatrix} 0 \\ -g \end{Bmatrix} \frac{\frac{2a}{3} \times \frac{ab}{2}}{3}$$

$$f^e = \begin{Bmatrix} 0 \\ -g \end{Bmatrix} \frac{a^2 b}{9}$$