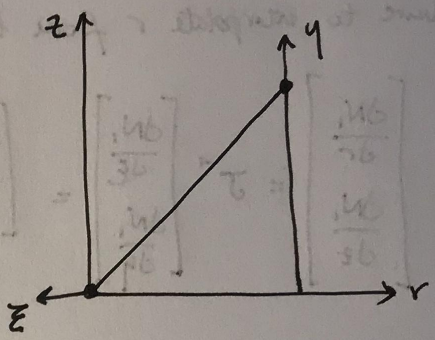
Computational Structural Mechanics and Dynamics

Oriol Call Piñol

Assignment 4: Structures of Revolution

$$\begin{cases} r_1 = 0 \\ r_2 = r_3 = a \end{cases} \qquad \begin{cases} z_1 = z_2 = 0 \\ z_3 = b \end{cases}$$

Isotropic material with $\nu = 0$



The stiffness matrix of an axisymmetric triangle is:

$$\boxed{K_e = 2\pi \int_A r B^T C B \, dA}$$

Using natural coordinates, the stiffness matrix takes this expression:

$$\boxed{k^e = 2\pi \int_0^1 \int_0^{1-\eta} r B^T C B \, |J| \, d\xi \, d\eta}$$

with $B = D^* N$, $N_2 = \xi = \xi_2$, $N_1 = 1 - \xi - \eta = \xi_1$, $N_3 = \eta = \xi_3$

$$D^* = \begin{bmatrix} \partial/\partial r & 0 \\ 0 & \partial/\partial z \\ 1 & 0 \\ \partial/\partial z & \partial/\partial r \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix} = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_1 & \xi_2 & \xi_3 \end{bmatrix}$$

$$C = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial r} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \frac{\partial N_3}{\partial z} \\ \frac{N_1}{r} & \frac{N_2}{r} & \frac{N_3}{r} & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \frac{\partial N_3}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial r} \end{bmatrix}$$

We have to interpolate r from the nodal coordinates: $\boxed{r = \sum r_i N_i = a(\xi+\eta) = a(\xi_2 + \xi_3)}$

① $\begin{bmatrix} \frac{\partial N_1}{\partial r} \\ \frac{\partial N_1}{\partial z} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -\frac{1}{a} \\ 0 \end{bmatrix}$ with $J = \sum \begin{bmatrix} \frac{\partial N_i}{\partial \xi} r_i & \frac{\partial N_i}{\partial \xi} z_i \\ \frac{\partial N_i}{\partial \eta} r_i & \frac{\partial N_i}{\partial \eta} z_i \end{bmatrix} = \begin{bmatrix} a & 0 \\ a & b \end{bmatrix}$

thus $J^{-1} = \frac{1}{ab} \begin{bmatrix} b & -a \\ 0 & a \end{bmatrix}$

② $\begin{bmatrix} \frac{\partial N_2}{\partial r} \\ \frac{\partial N_2}{\partial z} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial N_2}{\partial \xi} \\ \frac{\partial N_2}{\partial \eta} \end{bmatrix} = \begin{bmatrix} 1/a \\ -1/b \end{bmatrix}$

③ $\begin{bmatrix} \frac{\partial N_3}{\partial r} \\ \frac{\partial N_3}{\partial z} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial N_3}{\partial \xi} \\ \frac{\partial N_3}{\partial \eta} \end{bmatrix} = \begin{bmatrix} 0 \\ 1/b \end{bmatrix}$

Substituting in B we get:

$$B = \begin{bmatrix} -1/a & 1/a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/b & 1/b \\ \frac{1-\xi-\eta}{a(\xi+\eta)} & \frac{\xi}{a(\xi+\eta)} & \frac{\eta}{a(\xi+\eta)} & 0 & 0 & 0 \\ 0 & -1/b & 1/b & -1/a & 1/a & 0 \end{bmatrix}$$

We can substitute

$$\begin{cases} 1-\xi-\eta = \xi_1 \\ \xi = \xi_2 \\ \eta = \xi_3 \end{cases}$$

Using the formula to calculate the stifness matrix for a triangular element with p integration points:

$$\boxed{K^e = \sum \sum 2\pi w_k w_l B^T(\xi_k \eta_l) E B(\xi_k \eta_l) r(\xi_k \eta_l) J(\xi_k \eta_l)}$$

having $\begin{cases} w_k = w_l = 1/2 \\ \xi_1 = \xi_2 = \xi_3 = 1/3 \end{cases}$

by using Gauss 1 point integration we get:

$$K = \frac{\pi a b E}{2}\begin{bmatrix} 5/6a & -1/2a & 1/6a & 0 & 0 & 0 \\ -1/2a & \frac{2a}{3}\left(\frac{5}{4a^2}+\frac{1}{2b^2}\right) & \frac{2a}{3}\left(\frac{1}{4a^2}-\frac{1}{2b^2}\right) & 1/3b & -1/3b & 0 \\ 1/6a & \frac{2a}{3}\left(\frac{1}{4a^2}-\frac{1}{2b^2}\right) & \frac{2a}{3}\left(\frac{1}{4a^2}+\frac{1}{2b^2}\right) & -1/3b & 1/3b & 0 \\ 0 & 1/3b & -1/3b & 1/3a & -1/3a & 0 \\ 0 & -1/3b & 1/3b & -1/3a & \frac{2a}{3}\left(\frac{1}{2a^2}+\frac{1}{b^2}\right) & -2a/3b \\ 0 & 0 & 0 & 0 & -2a/3b^2 & 2a/3b \end{bmatrix}$$

2 Rows 2, 4, 6 are associated to the z-component, in this case, there is no rigid body motion, in the z axis, this is why the sum of those rows = 0.

In rows 1, 3, 5 doesn't happen, this is because this rows are related to the r-component of the system.

③ f^e with $b = [0, -g]^T$

$$f^e_{ext} = \sum \sum 2\pi \omega_k \omega_e N^T(\xi_k \eta_e)\, b(\xi_k \eta_e)\, r(\xi_k \eta_e)\, J(\xi_k \eta_e)$$

① ①

From previous calculations, we already have all the components of this formula.
After computing the information we get:

$$f^e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2ag/9 \\ -2ag/9 \\ -2ag/9 \end{bmatrix}$$