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Master of Science in Computational Mechanics 2018

Computational Structural Mechanics and Dynamics

"Structures of revolution"

Assignment 4.1

1. Compute the entries of $\mathbf{K}^e$ for the following axisymmetric triangle:

$$r_1 = 0, \quad r_2 = r_3 = a, \quad z_1 = z_2 = 0, \quad z_3 = b$$

The material is isotropic with $\nu = 0$ for which the stress-strain matrix is,

$$\boldsymbol{E} = E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

2. Show that the sum of the rows (and columns) 2, 4 and 6 of $\mathbf{K}^e$ must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.

3. Compute the consistent force vector $\mathbf{f}^e$ for gravity forces $\mathbf{b} = [0, -g]^T$.

Date of Assignment: 26 / 02 / 2018
Date of Submission: 5 / 03 / 2018

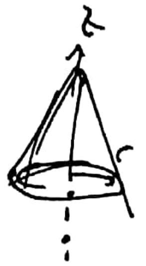
The assignment must be submitted as a pdf file named **As4-Surname.pdf** to the CIMNE virtual center.

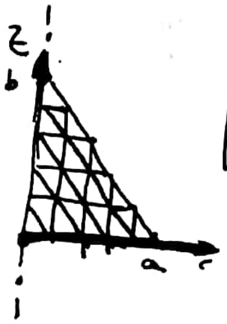
Computational Structural Mech & Dyn. Marcos Boniquet

Assigment 4.1

Axisymmetric triangle

1 – $\begin{cases} r_1 = 0 \\ r_2 = r_3 = a \\ z_1 = z_2 = 0 \\ z_3 = b \end{cases}$





isotropic

$$E = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{12} & E_{22} & E_{23} & E_{24} \\ E_{13} & E_{23} & E_{33} & 0 \\ E_{14} & E_{24} & 0 & E_{44} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \overset{\nu=0}{=} E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$$\begin{cases} E_{11} = E_{22} = E_{33} = 1 \\ E_{44} = 1/2 \end{cases}$$

$$N = \begin{bmatrix} N_1^e & N_2^e & \dots & N_n^e & 0 & 0 & 0 \\ 0 & 0 & & 0 & N_1^e & N_2^e & \dots N_n^e \end{bmatrix}$$

$$B = DN$$

$$D = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix}$$

3 node triangle element

$$B = \begin{bmatrix} q_r & 0 \\ 0 & q_z \\ q_\theta & 0 \\ q_z & q_r \end{bmatrix} \begin{cases} q_r = \left[\frac{\partial N_1^e}{\partial r} \; \frac{\partial N_2^e}{\partial r} \; \frac{\partial N_3^e}{\partial r} \; \dots \; \frac{\partial N_n^e}{\partial r} \right] = \left[\frac{\partial N_1^e}{\partial r}, \frac{\partial N_2^e}{\partial r}, \frac{\partial N_3^e}{\partial r} \right] \\ q_z = \left[\frac{\partial N_1^e}{\partial z} \; \frac{\partial N_2^e}{\partial z} \; \frac{\partial N_3^e}{\partial z} \; \dots \; \frac{\partial N_n^e}{\partial z} \right] = \left[\frac{\partial N_1^e}{\partial z}, \frac{\partial N_2^e}{\partial z}, \frac{\partial N_3^e}{\partial z} \right] \\ q_\theta = \left[\frac{N_1^e}{r} \; \frac{N_2^e}{r} \; \frac{N_3^e}{r} \dots \frac{N_n^e}{r} \right] = \left[\frac{N_1^e}{r}, \frac{N_2^e}{r}, \frac{N_3^e}{r} \right] \end{cases}$$

3x3 each

vertical & horiz. for each node of the element

$$EB = \begin{bmatrix} q_r & 0 \\ 0 & q_z \\ q_\theta & 0 \\ \frac{q_z}{2} & \frac{q_r}{2} \end{bmatrix} = \begin{bmatrix} C_{1r} & C_{1z} \\ C_{2r} & C_{2z} \\ C_{3r} & C_{3z} \\ C_{4r} & C_{4z} \end{bmatrix} \qquad B^T E B^{(e)} = \begin{bmatrix} S_{rr} & S_{rz} \\ S_{zr} & S_{zz} \end{bmatrix} \boxed{(6\times 6)} \text{ Matrix}$$

$$\begin{cases} S_{rr} = q_r^T C_{1r} + q_\theta^T C_{3r} + q_z^T C_{4r} \\ S_{rz} = S_{zr}^T = q_r^T C_{1z} + q_\theta^T C_{3z} + q_z^T C_{4z} \\ S_{zz} = q_r^T C_{4z} + q_z^T C_{2z} \end{cases} \longrightarrow \begin{cases} E \cdot S_{rr}^{(e)} = q_r^T q_r + q_\theta^T q_\theta + q_z^T \cdot \frac{q_z}{2} \\ E \cdot S_{rz}^{(e)} = S_{zr}^T = q_z^T \cdot \frac{q_r}{2} \\ E \cdot S_{zz}^{(e)} = q_r^T \frac{q_r}{2} + q_z^T q_z \end{cases}$$

$$\boxed{E \cdot S_{rr}^{(e)}} = \begin{pmatrix} \frac{\partial N_1}{\partial r} \\ \frac{\partial N_2}{\partial r} \\ \frac{\partial N_3}{\partial r} \end{pmatrix} \left(\frac{\partial N_1}{\partial r} \; \frac{\partial N_2}{\partial r} \; \frac{\partial N_3}{\partial r} \right) + \begin{pmatrix} \frac{N_1}{r} \\ \frac{N_2}{r} \\ \frac{N_3}{r} \end{pmatrix} \left(\frac{N_1}{r} \; \frac{N_2}{r} \; \frac{N_3}{r} \right) + \begin{pmatrix} \frac{\partial N_1}{\partial z} \\ \frac{\partial N_2}{\partial z} \\ \frac{\partial N_3}{\partial z} \end{pmatrix} \left(\frac{\partial N_1}{\partial z} \; \frac{\partial N_2}{\partial z} \; \frac{\partial N_3}{\partial z} \right) =$$

$$= \begin{bmatrix} \left(\frac{\partial N_1}{\partial r}\right)^2 & \frac{\partial N_1}{\partial r}\frac{\partial N_2}{\partial r} & \frac{\partial N_1}{\partial r}\frac{\partial N_3}{\partial r} \\ \frac{\partial N_2}{\partial r}\frac{\partial N_1}{\partial r} & \left(\frac{\partial N_2}{\partial r}\right)^2 & \frac{\partial N_2}{\partial r}\frac{\partial N_3}{\partial r} \\ \frac{\partial N_3}{\partial r}\frac{\partial N_1}{\partial r} & \frac{\partial N_3}{\partial r}\frac{\partial N_2}{\partial r} & \left(\frac{\partial N_3}{\partial r}\right)^2 \end{bmatrix} + \begin{bmatrix} \left(\frac{N_1}{r}\right)^2 & \frac{N_1 N_2}{r^2} & \frac{N_1 N_3}{r^2} \\ \frac{N_2 N_1}{r^2} & \left(\frac{N_2}{r}\right)^2 & \frac{N_2 N_3}{r^2} \\ \frac{N_3 N_1}{r^2} & \frac{N_3 N_2}{r^2} & \left(\frac{N_3}{r}\right)^2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \left(\frac{\partial N_1}{\partial z}\right)^2 & \frac{\partial N_1 \partial N_2}{\partial z \partial z} & \frac{\partial N_1}{\partial z}\frac{\partial N_3}{\partial z} \\ \frac{\partial N_2}{\partial z}\frac{\partial N_1}{\partial z} & \left(\frac{\partial N_2}{\partial z}\right)^2 & \frac{\partial N_2}{\partial z}\frac{\partial N_3}{\partial z} \\ \frac{\partial N_3}{\partial z}\frac{\partial N_1}{\partial z} & \frac{\partial N_3}{\partial z}\frac{\partial N_2}{\partial z} & \left(\frac{\partial N_3}{\partial z}\right)^2 \end{bmatrix}$$

$$E\,S_{rz}^{(e)} = \frac{1}{2}\begin{pmatrix} \frac{\partial N_1}{\partial z} \\ \frac{\partial N_2}{\partial z} \\ \frac{\partial N_3}{\partial z}\end{pmatrix}\begin{pmatrix} \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial r}\end{pmatrix} = \frac{1}{2}\begin{bmatrix} \frac{\partial N_1}{\partial z}\frac{\partial N_1}{\partial r} & \frac{\partial N_1}{\partial z}\frac{\partial N_2}{\partial r} & \frac{\partial N_1}{\partial z}\frac{\partial N_3}{\partial r} \\ \frac{\partial N_2}{\partial z}\frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z}\frac{\partial N_2}{\partial r} & \frac{\partial N_2}{\partial z}\frac{\partial N_3}{\partial r} \\ \frac{\partial N_3}{\partial z}\frac{\partial N_1}{\partial r} & \frac{\partial N_3}{\partial z}\frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z}\frac{\partial N_3}{\partial r} \end{bmatrix}$$

$$\| \quad E\,S_{zr}^{T}$$

$$E\,S_{zz}^{(e)} = \frac{1}{2}\begin{bmatrix} \left(\frac{\partial N_1}{\partial r}\right)^2 & \frac{\partial N_1}{\partial r}\frac{\partial N_2}{\partial r} & \frac{\partial N_1}{\partial r}\frac{\partial N_3}{\partial r} \\ \frac{\partial N_2}{\partial r}\frac{\partial N_1}{\partial r} & \left(\frac{\partial N_2}{\partial r}\right)^2 & \frac{\partial N_2}{\partial r}\frac{\partial N_3}{\partial r} \\ \frac{\partial N_3}{\partial r}\frac{\partial N_1}{\partial r} & \frac{\partial N_3}{\partial r}\frac{\partial N_2}{\partial r} & \left(\frac{\partial N_3}{\partial r}\right)^2 \end{bmatrix} + \begin{bmatrix} \left(\frac{\partial N_1}{\partial z}\right)^2 & \frac{\partial N_1}{\partial z}\frac{\partial N_2}{\partial z} & \frac{\partial N_1}{\partial z}\frac{\partial N_3}{\partial z} \\ \frac{\partial N_2}{\partial z}\frac{\partial N_1}{\partial z} & \left(\frac{\partial N_2}{\partial z}\right)^2 & \frac{\partial N_2}{\partial z}\frac{\partial N_3}{\partial z} \\ \frac{\partial N_3}{\partial z}\frac{\partial N_1}{\partial z} & \frac{\partial N_3}{\partial z}\frac{\partial N_2}{\partial z} & \left(\frac{\partial N_3}{\partial z}\right)^2 \end{bmatrix}$$

$$\begin{array}{cccccccc} & & & 2 & & 4 & & 6 \\ B^T E B^{(e)} = E & \Bigg[& \begin{matrix} \left(\frac{\partial N_1}{\partial r}\right)^2+\left(\frac{N_1}{r}\right)^2+\frac{1}{2}\left(\frac{\partial N_1}{\partial z}\right)^2 & \frac{\partial N_1}{\partial r}\frac{\partial N_2}{\partial r}+\frac{N_1N_2}{r^2}+\frac{1}{2}\frac{\partial N_1}{\partial z}\frac{\partial N_2}{\partial z} & \frac{\partial N_1}{\partial r}\frac{\partial N_3}{\partial r}+\frac{N_1N_3}{r^2}+\frac{1}{2}\frac{\partial N_1}{\partial z}\frac{\partial N_3}{\partial z} & \frac{1}{2}\frac{\partial N_1}{\partial z}\frac{\partial N_1}{\partial r} & \frac{1}{2}\frac{\partial N_1}{\partial z}\frac{\partial N_2}{\partial r} & \frac{1}{2}\frac{\partial N_1}{\partial z}\frac{\partial N_3}{\partial r} \\ \frac{\partial N_2}{\partial r}\frac{\partial N_1}{\partial r}+\frac{N_2N_1}{r^2}+\frac{1}{2}\frac{\partial N_2}{\partial z}\frac{\partial N_1}{\partial z} & \left(\frac{\partial N_2}{\partial r}\right)^2+\left(\frac{N_2}{r}\right)^2+\frac{1}{2}\left(\frac{\partial N_2}{\partial z}\right)^2 & \frac{\partial N_2}{\partial r}\frac{\partial N_3}{\partial r}+\frac{N_2N_3}{r^2}+\frac{1}{2}\frac{\partial N_2}{\partial z}\frac{\partial N_3}{\partial z} & \frac{1}{2}\frac{\partial N_2}{\partial z}\frac{\partial N_1}{\partial r} & \frac{1}{2}\frac{\partial N_2}{\partial z}\frac{\partial N_2}{\partial r} & \frac{1}{2}\frac{\partial N_2}{\partial z}\frac{\partial N_3}{\partial r} \\ \frac{\partial N_3}{\partial r}\frac{\partial N_1}{\partial r}+\frac{N_3N_1}{r^2}+\frac{1}{2}\frac{\partial N_3}{\partial z}\frac{\partial N_1}{\partial z} & \frac{\partial N_3}{\partial r}\frac{\partial N_2}{\partial r}+\frac{N_3N_2}{r^2}+\frac{1}{2}\frac{\partial N_3}{\partial z}\frac{\partial N_2}{\partial z} & \left(\frac{\partial N_3}{\partial r}\right)^2+\left(\frac{N_3}{r}\right)^2+\frac{1}{2}\left(\frac{\partial N_3}{\partial z}\right)^2 & \frac{1}{2}\frac{\partial N_3}{\partial z}\frac{\partial N_1}{\partial r} & \frac{1}{2}\frac{\partial N_3}{\partial z}\frac{\partial N_2}{\partial r} & \frac{1}{2}\frac{\partial N_3}{\partial z}\frac{\partial N_3}{\partial r} \\ \frac{1}{2}\frac{\partial N_1}{\partial z}\frac{\partial N_1}{\partial r} & \frac{1}{2}\frac{\partial N_2}{\partial z}\frac{\partial N_1}{\partial r} & \frac{1}{2}\frac{\partial N_1}{\partial z}\frac{\partial N_3}{\partial z} & \frac{1}{2}\left(\frac{\partial N_1}{\partial r}\right)^2+\left(\frac{\partial N_1}{\partial z}\right)^2 & \frac{1}{2}\frac{\partial N_1}{\partial r}\frac{\partial N_2}{\partial r}+\frac{\partial N_1}{\partial z}\frac{\partial N_2}{\partial z} & \frac{1}{2}\frac{\partial N_1}{\partial r}\frac{\partial N_3}{\partial r}+\frac{\partial N_1}{\partial z}\frac{\partial N_3}{\partial z} \\ \frac{1}{2}\frac{\partial N_1}{\partial z}\frac{\partial N_2}{\partial r} & \frac{1}{2}\frac{\partial N_2}{\partial z}\frac{\partial N_2}{\partial r} & \frac{1}{2}\frac{\partial N_3}{\partial z}\frac{\partial N_2}{\partial r} & \frac{1}{2}\frac{\partial N_2}{\partial r}\frac{\partial N_1}{\partial r}+\frac{\partial N_2}{\partial z}\frac{\partial N_1}{\partial z} & \frac{1}{2}\left(\frac{\partial N_2}{\partial r}\right)^2+\left(\frac{\partial N_2}{\partial z}\right)^2 & \frac{1}{2}\frac{\partial N_2}{\partial r}\frac{\partial N_3}{\partial r}+\frac{\partial N_2}{\partial z}\frac{\partial N_3}{\partial z} \\ \frac{1}{2}\frac{\partial N_1}{\partial z}\frac{\partial N_3}{\partial z} & \frac{1}{2}\frac{\partial N_2}{\partial z}\frac{\partial N_3}{\partial r} & \frac{1}{2}\frac{\partial N_3}{\partial z}\frac{\partial N_3}{\partial r} & \frac{1}{2}\frac{\partial N_3}{\partial r}\frac{\partial N_1}{\partial r}+\frac{\partial N_3}{\partial z}\frac{\partial N_1}{\partial z} & \frac{1}{2}\frac{\partial N_2}{\partial r}\frac{\partial N_3}{\partial r}+\frac{\partial N_3}{\partial z}\frac{\partial N_2}{\partial z} & \frac{1}{2}\left(\frac{\partial N_3}{\partial r}\right)^2+\left(\frac{\partial N_3}{\partial z}\right)^2 \end{matrix} & \Bigg] \end{array}$$

if linear triangle element

Natural coordinates

$$\begin{cases} N_1 = \xi_1 = \xi \\ N_2 = \xi_2 = \eta \\ N_3 = \xi_3 = 1-(\xi-\eta) \end{cases}$$

isoparametric linear triangle

Natural coordinates:

$$\begin{cases} \frac{\partial N_1}{\partial r} = \frac{\partial \xi}{\partial r} & \frac{\partial N_1}{\partial z} = \frac{\partial \xi}{\partial z} \\ \frac{\partial N_2}{\partial r} = \frac{\partial \eta}{\partial r} & \frac{\partial N_2}{\partial z} = \frac{\partial \eta}{\partial z} \\ \frac{\partial N_3}{\partial r} = -\frac{\partial \xi}{\partial r}+\frac{\partial \eta}{\partial r} & \frac{\partial N_3}{\partial z} = \frac{\partial \xi}{\partial z}+\frac{\partial \eta}{\partial z} = \boxed{\frac{\partial \eta}{\partial z}-\frac{\partial \xi}{\partial z}} \end{cases}$$

$$\| \quad \boxed{\frac{\partial \eta}{\partial r}-\frac{\partial \xi}{\partial r}}$$

$$K^{(e)} = 2\pi\int_A B^T E B^{(e)}\, r\, dA$$

$$\frac{\partial \xi_i}{\partial x} = \frac{b_i}{2A^e} \qquad \begin{cases} b_i = y_j - y_k \\ c_i = x_k - x_j \end{cases}$$

$$\frac{\partial \xi_i}{\partial y} = \frac{c_i}{2A^e}$$

isoparametric coordinates

3 – Compute consistent force f^e

for $b = [0, -g]^T$

$$f^{(e)} = \int_A N^T b \, r \, dA = \int_A \begin{bmatrix} N_1^e & 0 \\ N_2^e & 0 \\ N_3^e & 0 \\ 0 & N_1^e \\ 0 & N_2^e \\ 0 & N_3^e \end{bmatrix} \begin{bmatrix} 0 \\ -g \end{bmatrix} r \, dA = \int_A \begin{bmatrix} 0 \\ 0 \\ 0 \\ -N_1^e g \\ -N_2^e g \\ -N_3^e g \end{bmatrix} r \, dA = -gr \int_A \begin{bmatrix} 0 \\ 0 \\ 0 \\ N_1^e \\ N_2^e \\ N_3^e \end{bmatrix} dA$$

$$N^e = \begin{bmatrix} N_1^e & N_2^e & N_3^e & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1^e & N_2^e & N_3^e \end{bmatrix}$$

$$f^{(e)} = -gr \begin{bmatrix} 0 \\ 0 \\ 0 \\ \int N_1^e dA \\ \int N_2^e dA \\ \int N_3^e dA \end{bmatrix} = -gr \begin{bmatrix} 0 \\ 0 \\ 0 \\ \int \xi_1 dA \\ \int \xi_2 dA \\ \int \xi_3 dA \end{bmatrix} = \boxed{-gr \frac{1}{3} A^{(e)} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$$

$A^{(e)} = \frac{a \cdot b}{2} \Big/ N$ number of elements.

degree 1

$$\frac{1}{A^e} \int_A F(\xi_1, \xi_2, \xi_3) \, d\Omega \simeq F\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$\int F(\xi_1, \xi_2, \xi_3) \, d\Omega \simeq A^e \left[\frac{1}{3}\right]$$