HW4:

4.1.1:

Given:

$r_1 = 0, r_2 = r_3 = a$

$$\boldsymbol{E} = E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

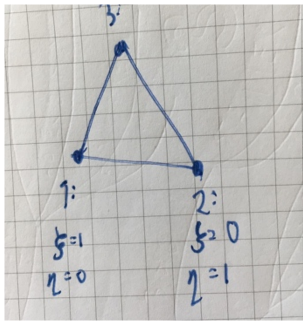
Searched:

$\boldsymbol{K}^e$

Solution:

The stiffness matrix for an axisymmetric triangle can be computed as

$$\boldsymbol{K}^e = \int_{-1}^{1}\int_{-1}^{1} h\boldsymbol{B}^T\boldsymbol{E}\boldsymbol{B}det\boldsymbol{J}d\xi d\eta \approx \sum_{i=1}^{p}\sum_{j}^{p} w_i w_j \boldsymbol{B}^T(\xi_i,\eta_j)\boldsymbol{E}\boldsymbol{B}(\xi_i,\eta_j)\mathrm{r}(\xi_i,\eta_j)J(\xi_i,\eta_j)$$



Using $w_i = w_j = 1$ we further obtain

$$\boldsymbol{K}^e \approx \sum_{i=1}^{p}\sum_{j}^{p} \boldsymbol{B}^T(\xi_i,\eta_j)\boldsymbol{E}\boldsymbol{B}(\xi_i,\eta_j)\mathrm{r}(\xi_i,\eta_j)\det\left(\boldsymbol{J}(\xi_i,\eta_j)\right)$$

For a triangle we chose the shape functions

$N_1 = \xi, \;\; N_2 = \eta, \;\;\; N_3 = 1 - (\xi + \eta) \;\; (1)$

and we express the coordinates in isoparametric system as follows:

$$r = N_1 r_1 + N_2 r_2 + N_3 r_3 = a - a\xi$$
$$z = N_1 z_1 + N_2 z_2 + N_3 z_3 = b - b\xi - b\eta. \quad (2)$$

Further it is true that

$$\boldsymbol{B} = \boldsymbol{D}\boldsymbol{N},$$

where

$$\boldsymbol{D} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix}$$

and

$$\boldsymbol{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

giving

$$\boldsymbol{B} = \boldsymbol{DN} = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} \end{bmatrix} = \{chain\ rule\}$$

$$= \begin{bmatrix} \frac{\partial N_1}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial N_1}{\partial \eta}\frac{\partial \eta}{\partial r} & 0 & \frac{\partial N_2}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial N_2}{\partial \eta}\frac{\partial \eta}{\partial r} & 0 & \frac{\partial N_3}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial N_3}{\partial \eta}\frac{\partial \eta}{\partial r} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial N_1}{\partial \eta}\frac{\partial \eta}{\partial z} & 0 & \frac{\partial N_2}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial N_2}{\partial \eta}\frac{\partial \eta}{\partial z} & 0 & \frac{\partial N_3}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial N_3}{\partial \eta}\frac{\partial \eta}{\partial z} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \\ \frac{\partial N_1}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial N_1}{\partial \eta}\frac{\partial \eta}{\partial z} & \frac{\partial N_1}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial N_1}{\partial \eta}\frac{\partial \eta}{\partial r} & \frac{\partial N_2}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial N_2}{\partial \eta}\frac{\partial \eta}{\partial z} & \frac{\partial N_2}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial N_2}{\partial \eta}\frac{\partial \eta}{\partial r} & \frac{\partial N_3}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial N_3}{\partial \eta}\frac{\partial \eta}{\partial z} & \frac{\partial N_3}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial N_3}{\partial \eta}\frac{\partial \eta}{\partial r} \end{bmatrix} =$$

$$= \{(1)\} = \begin{bmatrix} -\frac{1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b} & 0 & \frac{1}{b} \\ \frac{\xi}{a - a\xi} & 0 & \frac{\eta}{a - a\xi} & 0 & \frac{1 - (\xi + \eta)}{a - a\xi} & 0 \\ 0 & -\frac{1}{a} & -\frac{1}{b} & \frac{1}{a} & \frac{1}{b} & 0 \end{bmatrix}.$$

Using this result it follows that

$$\boldsymbol{B}^T \boldsymbol{E} \boldsymbol{B} = \begin{bmatrix} -\frac{1}{a} & 0 & \frac{\xi}{a - a\xi} & 0 \\ 0 & 0 & 0 & -\frac{1}{a} \\ \frac{1}{a} & 0 & \frac{\eta}{a - a\xi} & -\frac{1}{b} \\ 0 & -\frac{1}{b} & 0 & \frac{1}{a} \\ 0 & 0 & \frac{1 - (\xi + \eta)}{a - a\xi} & \frac{1}{b} \\ 0 & \frac{1}{b} & 0 & 0 \end{bmatrix} E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b} & 0 & \frac{1}{b} \\ \frac{\xi}{a - a\xi} & 0 & \frac{\eta}{a - a\xi} & 0 & \frac{1 - (\xi + \eta)}{a - a\xi} & 0 \\ 0 & -\frac{1}{a} & -\frac{1}{b} & \frac{1}{a} & \frac{1}{b} & 0 \end{bmatrix} =$$

$$= E\begin{bmatrix} \frac{1}{a^2}+\frac{\xi^2}{(a-a\xi)^2} & 0 & -\frac{1}{a^2}+\frac{\xi\eta}{(a-a\xi)^2} & 0 & \frac{\xi\big(1-(\xi+\eta)\big)}{(a-a\xi)^2} & 0 \\ 0 & \frac{1}{2a^2} & \frac{1}{2ab} & -\frac{1}{2a^2} & -\frac{1}{2ab} & 0 \\ -\frac{1}{a^2}+\frac{\xi\eta}{(a-a\xi)^2} & \frac{1}{2ab} & \frac{1}{a^2}+\frac{\eta^2}{(a-a\xi)^2}+\frac{1}{2b^2} & -\frac{1}{2ab} & \frac{\eta\big(1-(\xi+\eta)\big)}{(a-a\xi)^2}-1/2b^2 & 0 \\ 0 & -\frac{1}{2a^2} & -\frac{1}{2ab} & \frac{1}{b^2}+\frac{1}{2a^2} & \frac{1}{2ab} & -1/b^2 \\ \frac{\xi\big(1-(\xi+\eta)\big)}{(a-a\xi)^2} & -\frac{1}{2ab} & \frac{\eta\big(1-(\xi+\eta)\big)}{(a-a\xi)^2}-\frac{1}{2b^2} & \frac{1}{2ab} & \frac{\big(1-(\xi+\eta)\big)^2}{(a-a\xi)^2}+\frac{1}{2b^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{b^2} & 0 & 1/b^2 \end{bmatrix}$$

Now using

$$\boldsymbol{K}^e \approx \sum_{i=1}^{p}\sum_{j}^{p} \boldsymbol{B}^T(\xi_i,\eta_j)\boldsymbol{E}\boldsymbol{B}(\xi_i,\eta_j)\mathrm{r}(\xi_i,\eta_j)\ \det\big(\boldsymbol{J}(\xi_i,\eta_j)\big)$$

together with Gauss Quadrature rule degree 1 we get $\xi = \frac{1}{3},\ \eta = \frac{1}{3}$ which together with

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -a & -b \\ 0 & -b \end{bmatrix}$$

$$det\boldsymbol{J} = ab$$

and

$$r\left(\frac{1}{3},\frac{1}{3}\right) = \frac{2a}{3}$$

results in the stiffness matrix

$$\boldsymbol{K}^e \approx \frac{2Ea^2b}{3}\begin{bmatrix} \frac{1}{a^2}+\frac{1}{4a^2} & 0 & -\frac{1}{a^2}+\frac{1}{4a^2} & 0 & \frac{1}{4a^2} & 0 \\ 0 & \frac{1}{2a^2} & \frac{1}{2ab} & -\frac{1}{2a^2} & -\frac{1}{2ab} & 0 \\ -\frac{1}{a^2}+\frac{1}{4a^2} & \frac{1}{2ab} & \frac{1}{a^2}+\frac{1}{4a^2}+\frac{1}{2b^2} & -\frac{1}{2ab} & \frac{1}{4a^2}-\frac{1}{2b^2} & 0 \\ 0 & -\frac{1}{2a^2} & -\frac{1}{2ab} & \frac{1}{b^2}+\frac{1}{2a^2} & \frac{1}{2ab} & -\frac{1}{b^2} \\ \frac{1}{4a^2} & -\frac{1}{2ab} & \frac{1}{4a^2}-\frac{1}{2b^2} & \frac{1}{2ab} & \frac{1}{4a^2}+\frac{1}{2b^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{b^2} & 0 & \frac{1}{b^2} \end{bmatrix}$$

4.1.2:

Searched:

Show that sum of rows (and columns) 2,4 and 6 of $\boldsymbol{K}^e$ is zero and explain why.
Shot what sum of rows (and columns) 1,3 and 5 of $\boldsymbol{K}^e$ isn't zero and explain why.

Solution:

Since the stiffness matrix is symmetric the summation of rows are exactly the same as for the corresponding columns.

R/C 1: $\frac{1}{a^2}+\frac{1}{4a^2}-\frac{1}{a^2}+\frac{1}{4a^2}+\frac{1}{4a^2}=\frac{3}{4a^2}\neq 0$
R/C 2: $\frac{1}{2a^2}+\frac{1}{2ab}-\frac{1}{2a^2}-\frac{1}{2ab}=0$
R/C 3: $-\frac{1}{a^2}+\frac{1}{4a^2}+\frac{1}{2ab}+\frac{1}{a^2}+\frac{1}{4a^2}+\frac{1}{2b^2}-\frac{1}{2ab}+\frac{1}{4a^2}-\frac{1}{2b^2}=\frac{3}{4a^2}\neq 0$
R/C 4: $-\frac{1}{2a^2}-\frac{1}{2ab}+\frac{1}{b^2}+\frac{1}{2a^2}+\frac{1}{2ab}-\frac{1}{b^2}=0$
R/C 5: $\frac{1}{4a^2}-\frac{1}{2ab}+\frac{1}{4a^2}-\frac{1}{2b^2}+\frac{1}{2ab}+\frac{1}{4a^2}+\frac{1}{2b^2}=\frac{3}{4a^2}\neq 0$
R/C 6: $-\frac{1}{b^2}+\frac{1}{b^2}=0$

We can see that the sum of the rows/columns 2,4,6 is zero and for 1,3,5 not zero. The rows and columns containing the term "$\frac{N_i}{r}$" do not sum up to zero while the others do. This term causes singularity in the origin and has also been approximated for the case of axisymmetric triangle which explains why it doesn't perfectly sum up to zero.

4.1.3:

Given:

$$b=\begin{bmatrix}0\\-g\end{bmatrix}$$

Searched:

$\boldsymbol{f}_{ext}^{(e)}$

Solution:

The consistent force vector for axisymmetric elements is calculated as

$$\boldsymbol{f}_{ext}^{(e)}=\sum_{k=1}^{p}\sum_{l=1}^{p}w_k w_l \boldsymbol{N}^T(\xi_k,\eta_l)\boldsymbol{b}(\xi_k,\eta_l)r(\xi_k,\eta_l)\det(\boldsymbol{J}(\xi_k,\eta_l))$$

We have

$$\boldsymbol{N}=\begin{bmatrix}N_1 & 0 & N_2 & 0 & N_3 & 0\\ 0 & N_1 & 0 & N_2 & 0 & N_3\end{bmatrix},$$

$$N_1 = \xi,\ \ N_2 = \eta,\ \ \ N_3 = 1 - (\xi + \eta)$$

and

$$det\boldsymbol{J} = ab$$

which together with using $w_i = w_j = 1, r\left(\frac{1}{3},\frac{1}{3}\right) = \frac{2a}{3}$ and Gauss Quadrature in 1 degree gives

$$\boldsymbol{f}_{ext}^{(e)} = \frac{2a^2b}{3}\begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \\ 1/3 & 0 \\ 0 & 1/3 \\ 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}\begin{bmatrix} 0 \\ -g \end{bmatrix} = \frac{2a^2bg}{9}\begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

2:

Searched:

Shape functions $N_i, i = 1, \dots, 5$ for five node quadrilateral element.

Solution:

N_5 is calculated with the line-product method as follows:

$$N_5 = \alpha L_{1-2}L_{2-3}L_{3-4}L_{4-1} = (1+\eta)(1-\xi)(1-\eta)(1+\xi) = \alpha(1-\xi^2)(1-\eta^2).$$

We set $\alpha = 1$ at node 5 which via the rules for shape functions give

$$N_5 = 1, N_1 = N_2 = N_3 = N_4 = 0, \qquad \text{at node } 5.$$

For the corner nodes we have

$$\begin{aligned} N_1 &= C(1-\xi)(1-\eta)\xi\eta \\ N_2 &= C(1+\xi)(1-\eta)\xi\eta \\ N_3 &= C(1+\xi)(1+\eta)\xi\eta \\ N_4 &= C(1-\xi)(1+\eta)\xi\eta. \end{aligned}$$

With $C = \frac{1}{4}$ we achieve fulfilling the rule

$$\begin{aligned} N_i &= 1, \qquad \text{node } i \\ N_i &= 0, \qquad \text{node } j, j \neq i. \end{aligned}$$