Assignment 4

PRADEEP KUMAR BAL

March 5, 2018

PART 1:

Given that in a triangular element:$(r_1.z_1) = (0,0), (r_2, z_2) = (a, 0), (r_3, z_3) = (a, b)$.

$$\mathbf{E} = \; E * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}; \quad A = \tfrac{ab}{2}; \quad u^e = \begin{bmatrix} u_{r_1} & u_{z_1} & u_{r_2} & u_{z_2} & u_{r_3} & u_{z_3} \end{bmatrix}^T$$

;

The definition of iso-parametric in which the shape functions are the triangular co-ordinates: $N_1 = \zeta_1, \; N_2 = \zeta_2, \; N_3 = \zeta_3$:

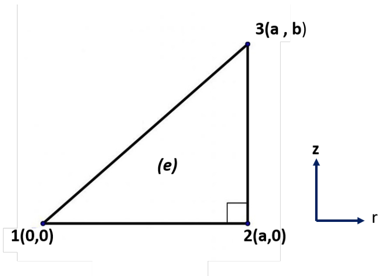

Figure 1: Given Element.

$$\begin{bmatrix} 1 & r & z & u_r & u_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \\ u_{r_1} & u_{r_2} & u_{r_3} \\ u_{z_1} & u_{z_2} & u_{z_3} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$$

Where,

$$\begin{bmatrix} 1 \\ r \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \; defines\ the\ element\ geometry$$

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} r_2 z_3 - r_3 z_2 & z_2 - z_3 & r_3 - r_2 \\ r_3 z_1 - r_1 z_3 & z_3 - z_1 & r_1 - r_3 \\ r_1 z_2 - r_2 z_1 & z_1 - z_2 & r_2 - r_1 \end{bmatrix} \begin{bmatrix} 1 \\ r \\ z \end{bmatrix}$$

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \frac{1}{ab} \begin{bmatrix} ab & -b & 0 \\ 0 & b & -a \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} 1 \\ r \\ z \end{bmatrix}$$

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 1 - \frac{r}{a} \\ \frac{r}{a} - \frac{z}{b} \\ \frac{z}{b} \end{bmatrix}$$

$$B^e = \frac{1}{2A} \begin{bmatrix} z_{23} & 0 & z_{31} & 0 & z_{12} & 0 \\ 0 & r_{32} & 0 & r_{13} & 0 & r_{21} \\ \frac{2A\ N_1}{r} & 0 & \frac{2A\ N_2}{r} & 0 & \frac{2A\ N_3}{r} & 0 \\ r_{32} & z_{23} & r_{13} & z_{31} & r_{21} & z_{12} \end{bmatrix}$$

$$B^e = \frac{1}{2A} \begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ \frac{2A\ N_1}{r} & 0 & \frac{2A\ N_2}{r} & 0 & \frac{2A\ N_3}{r} & 0 \\ 0 & -b & -a & b & a & 0 \end{bmatrix}$$

$$K^e = \int_{\Omega^e} r \quad B^{eT}\ \mathbf{E}\ B^e\ d\Omega$$

$$K^e = \frac{E}{4A^2} \int_{\Omega^e} \begin{bmatrix} b^2\ r + \frac{4A^2\ N_1\ N_1}{r} & 0 & -b^2\ r + \frac{4A^2\ N_1\ N_2}{r} & 0 & \frac{4A^2\ N_1\ N_3}{r} & 0 \\ 0 & \frac{b^2 r}{2} & \frac{abr}{2} & -\frac{b^2 r}{2} & -\frac{abr}{2} & 0 \\ -b^2 r\ + \frac{4A^2\ N_1\ N_2}{r} & -\frac{abr}{2} & b^2 r\ +\ \frac{4A^2\ N_2\ N_2}{r}\ +\ \frac{a^2 r}{2} & -\frac{abr}{2} & -\frac{a^2 r}{2}\ + \frac{4A^2\ N_2\ N_3}{r} & 0 \\ 0 & -\frac{b^2 r}{2} & -\frac{abr}{2} & a^2 r\ +\ \frac{b^2 r}{2} & \frac{abr}{2} & -a^2 r \\ \frac{4A^2\ N_1\ N_3}{r} & -\frac{abr}{2} & \frac{4A^2\ N_2\ N_3}{r} -\ \frac{a^2 r}{2} & \frac{abr}{2} & \frac{a^2 r}{2} + \frac{4A^2\ N_3\ N_3}{r} & 0 \\ 0 & 0 & 0 & -a^2 r & 0 & a^2 r \end{bmatrix} d\Omega$$

ANALYTICAL EXACT INTEGRATION OVER THE DOMAIN:-

$$K_{11} \ = \frac{E}{4A^2} \int_{\Omega^e} b^2\ r + \frac{4A^2\ N_1\ N_1}{r}\ d\Omega$$

$$= \frac{E}{a^2 b^2} \int_{r=0}^{r=a} \left[\int_{z=0}^{z=\frac{br}{a}} b^2\ r + \frac{4A^2\ N_1\ N_1}{r}\ dz \right] dr$$

$$= E \int_{r=0}^{r=a} \left[\int_{z=0}^{z=\frac{br}{a}} \{ \frac{r}{a^2} + \frac{N_1^2}{r}\ \} dz \right] dr\ ; \quad N_1 = 1 - \frac{r}{a}$$

$$= E \int_{r=0}^{r=a}\ \frac{br}{a} \{ \frac{r}{a^2} + \frac{N_1^2}{r}\ \}\ dr\ ; \quad N_1 = 1 - \frac{r}{a}$$

$$= E \int_{r=0}^{r=a} [\ \frac{2\ b\ r^2}{a^3} + \frac{b}{a} - \frac{2br}{a^2}]\ dr\ ;$$

$$= \ \frac{2Eb}{3}$$

$$K_{12} = 0$$

$$K_{13}\ = \frac{E}{4A^2} \int_{\Omega^e} -b^2\ r + \frac{4A^2\ N_1\ N_2}{r}\ d\Omega$$

$$= \frac{E}{a^2b^2} \int_{r=0}^{r=a} \left[\int_{z=0}^{z=\frac{br}{a}} -b^2\ r + \frac{4A^2\ N_1\ N_2}{r}\ dz \right] dr; \quad N_1 = 1 - \frac{r}{a}; \quad N_2 = \ \frac{r}{a} - \frac{z}{b}$$

$$= \frac{E}{a^2b^2} \int_{r=0}^{r=a} \left[\int_{z=0}^{z=\frac{br}{a}} -2b^2r + ab^2 + zab - \frac{a^2bz}{r}\ dz \right] dr;$$

$$= \frac{E}{a^2b^2} \int_{r=0}^{r=a} [\ -\frac{3\ b^3\ r^2}{a} + \frac{b^3r}{2}]\ dr\ ;$$

$$= -\frac{Eb}{4}$$

$$K_{14} = 0$$

$$K_{66} = \frac{Ea^2}{3b}$$

$$K_{15}\ = \frac{E}{4A^2} \int_{\Omega^e} \frac{4A^2\ N_1\ N_3}{r}\ d\Omega; \quad N_3 = \frac{z}{b}$$

$$= E \int_{r=0}^{r=a} \left[\int_{z=0}^{z=\frac{br}{a}} (\ 1 - \frac{r}{a}\ )\ \frac{z}{b}\ dz \right] dr;$$

$$= \frac{Eb}{2a^2} \int_{r=0}^{r=a} [\ r^2 - \frac{r^3}{a}]\ dr\ ;$$

$$= -\frac{Eab}{24}$$

$$K_{16} = 0$$

$$K_{22}\ = \frac{E}{4A^2} \int_{\Omega^e} \frac{b^2r}{2}\ d\Omega;$$

$$= \tfrac{E}{a^2b^2} \int_{r=0}^{r=a} [\ r^2\ \frac{b^3}{2a}]\ dr \qquad = \tfrac{Eb}{6}$$

$$K_{23}\ = \tfrac{E}{4A^2} \int_{\Omega^e} \frac{abr}{2}\ d\Omega;$$

$$= \tfrac{E}{2a^2} \int_{r=0}^{r=a} [\ r^2\ ]\ dr \qquad = \tfrac{Ea}{6}$$

$$K_{24} = -K_{22} = -\tfrac{Eb}{6}$$

$$K_{25} = -K_{23} = -\tfrac{Ea}{6}$$

$$K_{26} = 0$$

$$K_{33}\ = \tfrac{E}{4A^2} \int_{\Omega^e} [\ b^2r\ +\ \frac{4A^2\ N_2\ N_2}{r}\ +\ \frac{a^2r}{2}\ ]\ d\Omega;$$

$$= -\tfrac{K_{24}}{2} + \tfrac{K_{66}}{2} + E \int_{\Omega^e} [\ \frac{4A^2\ N_2\ N_2}{r}\ ]\ d\Omega;$$

$$= -\tfrac{K_{24}}{2} + \tfrac{K_{66}}{2}\ + E \int_{r=0}^{r=a} \left[\int_{z=0}^{z=\frac{br}{a}} [\ (\frac{r}{a^2}\ )\ + \frac{z^2}{rb^2}\ - \frac{2z}{ab}\ ]\ dz \right] dr;$$

$$= \tfrac{K_{66}}{2} - \tfrac{K_{24}}{2} + E \int_{r=0}^{r=a} [\ \frac{br^2}{3a^3}]\ dr$$

$$= \tfrac{7Eb}{36} + \tfrac{Ea^2}{6b}$$

$$K_{34} = -K_{23}$$

$$K_{35}\ = \tfrac{E}{4A^2} \int_{\Omega^e} [\ \frac{4A^2\ N_2\ N_3}{r}\ -\ \frac{a^2r}{2}\ ]\ d\Omega;$$

$$= -\tfrac{K_{66}}{2}\ + E \int_{r=0}^{r=a} \left[\int_{z=0}^{z=\frac{br}{a}} [\ (\frac{1}{r}\ )\ \frac{rb-az}{ab}\ \frac{z}{b}\ ]\ dz \right] dr;$$

$$= -\tfrac{K_{66}}{2} + E \int_{r=0}^{r=a} [\ \frac{br^2}{3a^3}]\ dr$$

$$K_{36} = 0$$

$$K_{44}\ = \tfrac{E}{4A^2} \int_{\Omega^e} [\ a^2 r\ +\ \frac{b^2 r}{2}\ ]\ d\Omega;$$

$$= K_{66} - K_{24} = \tfrac{Ea^2}{3b} + \tfrac{Eb}{6}$$

$$K_{45} = K_{23} = \tfrac{Ea}{6}$$

$$K_{46} = -K_{66} = -\tfrac{Ea^2}{3b}$$

$$K_{55}\ = \tfrac{E}{4A^2} \int_{\Omega^e} [\ \frac{4A^2\ N_3\ N_3}{r}\ +\ \frac{a^2 r}{2}\ ]\ d\Omega;$$

$$= \tfrac{K_{66}}{2}\ + E \int_{r=0}^{r=a} \left[\int_{z=0}^{z=\frac{br}{a}} [\ (\frac{E}{r}\ )\ \frac{z^2}{b^2}\ \ ]\ dz \right] dr;$$

$$= \tfrac{K_{66}}{2} + E \int_{r=0}^{r=a} [\ \frac{br^2}{3a^3}]\ dr$$

$$= \tfrac{Ea^2}{6b} + \tfrac{Eb}{9}$$

$$K_{56} = 0$$

$$K_{66}\ = \tfrac{E}{4A^2} \int_{\Omega^e} [\ a^2 r\ ]\ d\Omega;$$

$$= E \int_{r=0}^{r=a} [\ \frac{r^2}{ab}]\ dr$$

$$K_{66} = \tfrac{Ea^2}{3b}$$

$$\mathbf{K}^e = E \begin{bmatrix} \frac{2b}{3} & 0 & -\frac{b}{4} & 0 & \frac{ab}{24} & 0 \\ 0 & \frac{b}{6} & \frac{a}{6} & -\frac{b}{6} & -\frac{a}{6} & 0 \\ -\frac{b}{4} & \frac{a}{6} & \frac{7b}{36} + \frac{a^2}{6b} & -\frac{a}{6} & \frac{b}{18} - \frac{a^2}{6b} & 0 \\ 0 & -\frac{b}{6} & -\frac{a}{6} & \frac{b}{6} + \frac{a^2}{3b} & \frac{a}{6} & -\frac{a^2}{3b} \\ \frac{ab}{24} & -\frac{a}{6} & -\frac{a^2}{6b} + \frac{b}{18} & \frac{a}{6} & \frac{a^2}{6b} + \frac{b}{9} & 0 \\ 0 & 0 & 0 & -\frac{a^2}{3b} & 0 & \frac{a^2}{3b} \end{bmatrix}$$

PART 2:

(a) It can be observed from the above matrix: $Row\ (2) + Row\ (4) + Row\ (6)$

$$= \begin{bmatrix} 0 & \frac{b}{6} & \frac{a}{6} & -\frac{b}{6} & -\frac{a}{6} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{b}{6} & -\frac{a}{6} & (\frac{b}{6} + \frac{a^2}{3b}) & \frac{a}{6} & -\frac{a^2}{3b} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -\frac{a^2}{3b} & 0 & \frac{a^2}{3b} \end{bmatrix}$$

$$= \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It can be concluded that in the Z direction rigid body motion can be achieved. If we consider $u^e = \begin{bmatrix} u_{r_1} & u_{z_1} & u_{r_2} & u_{z_2} & u_{r_3} & u_{z_3} \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}^T$, We can get zero internal resistance forces to the prescribed motion, which depicts the rigid body motion in the Z direction can be retrieved.

(b) On the other hand $Row\ (1) + Row\ (3) + Row\ (5)$

$$= \begin{bmatrix} \frac{2b}{3} & 0 & -\frac{b}{4} & 0 & \frac{ab}{24} & 0 \end{bmatrix} + \begin{bmatrix} -\frac{b}{4} & \frac{a}{6} & \frac{7b}{36} + \frac{a^2}{6b} & -\frac{a}{6} & \frac{b}{18} - \frac{a^2}{6b} & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{ab}{24} & -\frac{a}{6} & -\frac{a^2}{6b} + \frac{b}{18} & \frac{a}{6} & \frac{a^2}{6b} + \frac{b}{9} & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It can be concluded that in the r direction it is not possible to achieve the rigid body motion. Only deformation happen in the direction of r.

PART 3:

$$f_{ext}^{(e)} = \int_{\Omega^e} [\ \mathbf{N}^T\ \mathbf{b}\ r\ ]\ d\Omega;$$

$$\mathbf{N}^e = \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & N_3^e & 0 \\ 0 & N_1^e & 0 & N_2^e & 0 & N_3^e \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 0 & -\rho\ g \end{bmatrix}^T$$

$$\begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix} = \begin{bmatrix} 1 - \frac{r}{a} \\ \frac{r}{a} - \frac{z}{b} \\ \frac{z}{b} \end{bmatrix}$$

$$f_{ext}^{(e)} = \int_{\Omega^e} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} 0 \\ -\rho\ g \end{bmatrix} \ r \ \ d\Omega;$$

$$f_{ext}^{(e)} = \int_{\Omega^e} \begin{bmatrix} 0 \\ -\rho\ grN_1 \\ 0 \\ -\rho\ grN_2 \\ 0 \\ -\rho\ grN_3 \end{bmatrix} \ d\Omega;$$

$$f_1 = 0$$

$$f_2 = \int_{\Omega^e} -\rho\ grN_1 \ \ d\Omega$$

$N_1 = 1 - \frac{r}{a};$

$$f_2 = -\rho\ g \int_{r=0}^{r=a} \left[\int_{z=0}^{z=\frac{br}{a}} [\ (\frac{ar - r^2}{a} \ ) \quad]\ dz \right] dr;$$

$$= -\rho\ g \int_{r=0}^{r=a} [\ \frac{abr^2 - br^3}{a^2}]\ dr$$

$$f_2 = -\frac{\rho\ ga^2 b}{12}$$

$$f_3 = 0$$

$$f_4 = \int_{\Omega^e} -\rho\ grN_2 \ \ d\Omega$$

$N_2 = \frac{r}{a} - \frac{z}{b};$

$$f_4 = -\rho\ g \int_{r=0}^{r=a} \left[\int_{z=0}^{z=\frac{br}{a}} [\ (\frac{r^2 b - zar}{ab} \ ) \quad]\ dz \right] dr;$$

$$= -\rho\ g \int_{r=0}^{r=a} [\ \frac{br^3}{2a^2}]\ dr$$

$$f_4 = -\frac{\rho\ ga^2 b}{8}$$

$$f_5 = 0$$

$$f_6 \ = \int_{\Omega^e} \ -\rho \ grN_3 \ \ d\Omega$$

$N_3 = \frac{z}{b};$

$$f_6 = -\rho \ g\int_{r=0}^{r=a} \left[\ \int_{z=0}^{z=\frac{br}{a}} [\ (\frac{zr}{b} \ ) \quad] \ dz \ \right] \ dr;$$

$$= -\rho \ g \ \int_{r=0}^{r=a} [\ \frac{br^3}{2a^2}] \ dr$$

$f_6 = -\frac{\rho \ ga^2 b}{8}$

$$f_{ext} = \begin{bmatrix} 0 \\ -\frac{\rho \ ga^2 b}{12} \\ 0 \\ -\frac{\rho \ ga^2 b}{8} \\ 0 \\ -\frac{\rho \ ga^2 b}{8} \end{bmatrix}$$

—-END—-