UNIVERSITAT POLITÈCNICA
DE CATALUNYA
BARCELONATECH

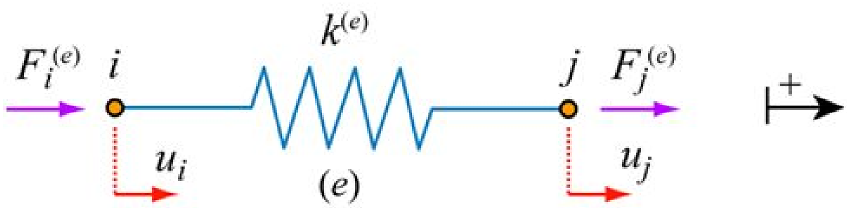
Assignment 4

Computational Structural Mechanics and Dynamics

Sebastian Ares de Parga Regalado

Master in Numerical Methods in Engineering
Universitat Politècnica de Catalunya

March 2020



Contents

On "Isoparametric representation"

1 Assignment 4.1

A 3-node straight bar element is defined by 3 nodes: 1, 2 and 3 with axial coordinates x_1, x_2 and x_3 respectively as illustrated in figure below. The element has axial rigidity EA, and length $l = x_1 – x_2$. The axial displacement is u(x). The 3 degrees of freedom are the axial node displacement u_1, u_2 and u_3. The isoparametric definition of the element is

$$\begin{bmatrix} 1 \\ x \\ u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

in which $N_1^e(\xi)$ are the shape functions of a three bar element. Node 3 lies between 1 and 2 but is not necessarily at the midpoint x = l/2. For convenience define,

$$x_1 = 0 \qquad x_2 = (\frac{1}{2} + \alpha)l \qquad x_3 = l$$

where $-\frac{1}{2} < \alpha < \frac{1}{2}$ characterizes the location of node 3 with respect to the element center. If $\alpha = 0$ node 3 is located at the midpoint between 1 and 2.

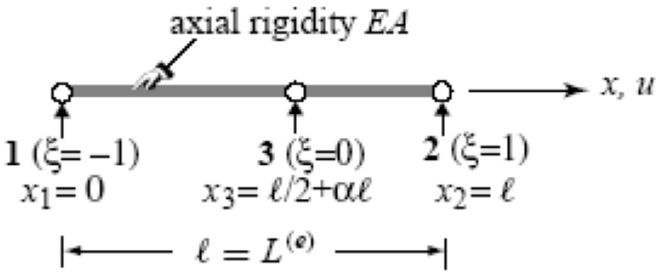


Figure 1: The three-node bar element in its local system

1. Get the Jacobian $J = \frac{dx}{d\xi}$ in terms of l, α and ξ. Show that,
   - if $-\frac{1}{4} < \alpha < \frac{1}{4}$ then $J > 0$ over the whole element $-1 \leq \xi \leq 1$.
   - if $\alpha = 0$, $J = \frac{l}{2}$ is a constant over the element.
2. Obtain the 1x3 strain displacement matrix B relating $e = \frac{du}{dx} = Bu^e$ where u^e is the column 3-vector of the node displacement u_1, u_2 and u_3. The entries of B are functions of l, α and ξ.

1.1 Solution

For a quadratic Lagrange element with three nodes at $\xi_1 = -1$, $\xi_2 = 0$ and $\xi = 1$ the shape functions are:

$$N_1 = \frac{1}{2}\xi(\xi - 1)$$

$$N_2 = (1 - \xi^2)$$

$$N_3 = \frac{1}{2}\xi(\xi + 1)$$

A parametric interpolation of the element geometry yields as follows:

$$x = \sum_{i=1}^{3} N_i(\xi) x_i$$

Substituting the shape functions and the values of x_i:

$$x = \frac{1}{2}\xi(\xi - 1)(0) + (1 - \xi^2)(\frac{1}{2} + \alpha)l + \frac{1}{2}\xi(\xi + 1)l$$

$$x = -\frac{l}{2}[2\xi^2\alpha - \xi - 2\alpha - 1]$$

Where the Jacobian is defined as:

$$J = \frac{dx}{d\xi} = -\frac{l}{2}[4\xi\alpha - 1]$$

To show that $J > 0$ when $-\frac{1}{4} < \alpha < \frac{1}{4}$ for the whole element $-1 \leq \xi \leq 1$, we have that:

$$|\alpha \cdot \xi| = |\alpha| \cdot |\xi| < \frac{|\xi|}{4} \leq \frac{1}{4}$$

$$|\alpha \cdot \xi| < \frac{1}{4}$$

$$\frac{-1}{4} < \alpha \cdot \xi < \frac{1}{4}$$

$$4\alpha \cdot \xi < 1$$

$$4\alpha \cdot -1 < 0$$

$$l(4\alpha \cdot -1) < 0$$

$$-l(4\alpha \cdot -1) > 0$$

$$-\frac{l}{2}(4\alpha \cdot -1) > 0$$

$$J > 0$$

If $\alpha = 0$, J doesn't depend on ξ, therefore, it is a constant:

$$J = -\frac{l}{2}[4\xi(0) - 1] = -\frac{l}{2}[-1]$$

$$J = \frac{l}{2}$$

The strain displacement matrix B is defined as follows:

$$B = \frac{dN}{dx} = J^{-1}\frac{dN}{d\xi}$$

Where:

$$J^{-1} = -\frac{2}{l(4\xi\alpha - 1)}$$

$$N = [N_1 \quad N_2 \quad N_3]$$

$$\frac{N_1}{\xi} = \frac{2\xi - 1}{2}$$
$$\frac{N_2}{\xi} = -2\xi$$
$$\frac{N_3}{\xi} = \frac{2\xi + 1}{2}$$

Therefore:

$$B = -\frac{2}{l(4\xi\alpha - 1)} \begin{bmatrix} \frac{2\xi-1}{2} & -2\xi & \frac{2\xi+1}{2} \end{bmatrix}$$
$$B = \begin{bmatrix} -\frac{2\xi-1}{l(4\xi\alpha-1)} & \frac{4\xi}{l(4\xi\alpha-1)} & -\frac{2\xi+1}{l(4\xi\alpha-1)} \end{bmatrix}$$
$$B = \frac{1}{l(4\xi\alpha - 1)} \begin{bmatrix} -(2\xi - 1) & 4\xi & -(2\xi + 1) \end{bmatrix}$$

On "Structures of revolution"

2 Assignment 4.2

1. Compute the entries of K^e for the following axisymmetric triangle:

$$r_1 = 0 \qquad r_2 = r_3 = a, \qquad z_1 = z_2 = 0, \qquad z_3 = b$$

   The material is isotropic with $\nu = 0$ for which the stress-strain matrix is,

$$E = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

2. Show that the sum of the rows (and columns) 2, 4 and 6 of K^e must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.

3. Compute the consistent force vector f^e for gravity forces $b = [0, -g]^T$.

2.1 Solution

The stiffness matrix can be computed according to the general relationship:

$$K_{ij}^e = 2\pi \int B_i^T D B_j r dr dz$$

Since the integration over the element is algebraically complex, the numerical integration fits better this case, evaluating all quantities for a centroidal point:

$$\bar{r} = \frac{r_i + r_j + r_m}{3} \qquad and \qquad \bar{z} = \frac{z_i + z_j + z_m}{3}$$

In this case the stiffness matrix approximation is defined as follows:

$$K_{ij}^e = 2\pi \bar{B}_i^T D \bar{B}_j \bar{r} A$$

Where the shape functions are and its derivatives are:

$$N_1 = 1 - \frac{r}{a}; \qquad \frac{\partial N_1}{\partial r} = -\frac{1}{a}; \qquad \frac{\partial N_1}{\partial z} = 0$$
$$N_2 = \frac{r}{a} - \frac{z}{b}; \qquad \frac{\partial N_2}{\partial r} = \frac{1}{a}; \qquad \frac{\partial N_2}{\partial z} = -\frac{1}{b}$$
$$N_3 = \frac{z}{b}; \qquad \frac{\partial N_3}{\partial r} = 0; \qquad \frac{\partial N_3}{\partial z} = \frac{1}{b}$$

Note: The values of "r" and "z" will have to be substituted by $\bar{r}$ and $\bar{z}$ respectively.

Computing the centroide of the element, we get:

$$\bar{r} = \frac{2a}{3}; \qquad \bar{z} = \frac{b}{3}$$

The strain matrix is defined as follows:

$$B_i = \begin{bmatrix} \frac{\partial N_i}{\partial r} & 0 \\ 0 & \frac{\partial N_i}{\partial z} \\ \frac{N_i}{r} & 0 \\ \frac{\partial N_i}{\partial r} & \frac{\partial N_i}{\partial r} \end{bmatrix}$$

Substituting, the following strain matrix is obtained:

$$B = \begin{bmatrix} -1/a & 0 & 1/a & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/b & 0 & 1/b \\ 1/(2*a) & 0 & 1/(2*a) & 0 & 1/(2*a) & 0 \\ 0 & -1/a & -1/b & 1/a & 1/b & 0 \end{bmatrix}$$

Poisson ratio is considered to be 0, therefore the constitutive matrix yields:

$$D = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

The area of the element is:

$$A = \frac{1}{2} det(\begin{bmatrix} 1 & 0 & 0 \\ 1 & a & 0 \\ 1 & a & b \end{bmatrix}) = \frac{ab}{2}$$

Now the stiffness matrix can be obtained:

$$K_{ij}^e = 2\pi \frac{2a}{3} \frac{ab}{2} \bar{B}_i^T D \bar{B}_j$$

$$K_{ij}^e = \frac{2\pi a^2 b}{3} \bar{B}_i^T D \bar{B}_j$$

$$K_{ij}^e = \frac{2\pi a^2 b}{3} E \begin{bmatrix} \frac{5}{4a^2} & 0 & \frac{-3}{4a^2} & 0 & \frac{1}{4a^2} & 0 \\ 0 & \frac{1}{2a^2} & \frac{1}{2ab} & \frac{-1}{2a^2} & \frac{-1}{2ab} & 0 \\ \frac{-3}{4a^2} & \frac{1}{2ab} & \frac{5}{4a^2} + \frac{1}{2b^2} & \frac{-1}{2ab} & \frac{1}{4a^2} - \frac{1}{2b^2} & 0 \\ 0 & \frac{-1}{2a^2} & \frac{-1}{2ab} & \frac{1}{2a^2} + \frac{1}{b^2} & \frac{1}{2ab} & \frac{-1}{b^2} \\ \frac{1}{4a^2} & \frac{-1}{2ab} & \frac{1}{4a^2} - \frac{1}{2b^2} & \frac{1}{2ab} & \frac{1}{4a^2} + \frac{1}{2b^2} & 0 \\ 0 & 0 & 0 & \frac{-1}{b^2} & 0 & \frac{1}{b^2} \end{bmatrix}$$

A first approximation if the body forces are constant is:

$$f_i^e = -2\pi \mathbf{b} \frac{\bar{r}A}{3}$$

$$f_i^e = -2\pi \mathbf{b} \frac{1}{3} \frac{2a}{3} \frac{ab}{2} = -2\pi \frac{a^2 b}{9} \mathbf{b}$$

Where $\mathbf{b}$, is:

$$\mathbf{b} = \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

Therefore:

$$f = 2\pi \frac{a^2 bg}{9} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

3 Discussion

The stiffness matrix and consistent force vector was obtained through a centroidal approximation to avoid the complexity of the analytical integration, but it could have been done through analytical or numerical approximation as well.
Once the stiffness matrix is obtained, it can be seen that the sum of the rows (and columns) 2, 4 and 6 vanishes, while the sum of the rows (and columns) 1,3 and 5 does not vanish. This is because in the "z" direction there must be a balance for each axisimetry plane, while the balance in the "r" axis direction will occur with the geometric counterpart of the problem.
The gravity forces vector $\mathbf{b}$, has to be multiplied by the density of the material, therefore the consistent forces vector yields as follows:

$$f = 2\pi \frac{a^2 b\rho g}{9} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$