1.

Assignment-3.1

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Computational Mechanics

Computational Structural Mechanics and Dynamics

1. $x_1 = 0$, $y_1 = 0$, $x_2 = 3$, $y_2 = 1$, $x_3 = 2$, $y_3 = 2$

$$E = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}, \quad h = 1$$

For linear triangle Element, Element stiffness matrix

$$K^{(e)} = \int_{\Omega^{(e)}} h\, B^T E B \, d\Omega = B^T E B \int_{\Omega^{(e)}} h \, d\Omega$$

$y_{23} = y_2 - y_3 = 1-2, = \underline{-1}$; $x_{32} = 2-3 = \underline{-1}$; $y_{31} = 2-0 = \underline{2}$; $x_{13} = \underline{-2}$

$y_{12} = 0-1 = \underline{-1}$; $x_{21} = 3-0 = \underline{3}$

$$B = \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix}$$

$$B^T = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \\ -1 & 0 & 3 \\ 0 & 3 & -1 \end{bmatrix}$$

$$K^{(e)} = \frac{1}{4A} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \\ -1 & 0 & 3 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix}$$

where $A = \frac{1}{2} \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix} = \frac{1}{2}[6-2] = 2$

$$K^{(e)} = \frac{1}{8} \begin{bmatrix} -100 & -25 & -50 \\ -25 & -100 & -50 \\ 200 & 50 & -100 \\ -50 & -200 & 100 \\ -100 & -25 & 150 \\ 75 & 300 & -50 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix}$$

$$K = \frac{1}{8}\begin{bmatrix} 150 & 75 & -100 & -50 & -50 & -25 \\ 75 & 150 & 50 & 100 & -125 & -250 \\ -100 & 50 & 600 & -300 & -500 & 250 \\ -50 & 100 & -300 & 600 & 350 & -700 \\ -50 & -125 & -500 & 350 & 550 & -225 \\ -25 & -250 & 250 & -700 & -225 & 950 \end{bmatrix}$$

$$K = \begin{bmatrix} 18.75 & 9.375 & -12.5 & -6.25 & -6.25 & -3.125 \\ 9.375 & 18.75 & 6.25 & 12.5 & -15.625 & -31.25 \\ -12.5 & 6.25 & 75 & -37.5 & -62.5 & 31.25 \\ -6.25 & 12.5 & -37.5 & 75 & 43.75 & -87.5 \\ -6.25 & -15.625 & -62.5 & 43.75 & 68.75 & -28.125 \\ -3.125 & -31.25 & 31.25 & -87.5 & -28.125 & 118.75 \end{bmatrix}$$

2. Rows (1, 3, 5)

Row (1) + Row (3) + Row (5)

$$= [18.75 \quad 9.375 \quad -12.5 \quad -6.25 \quad -6.25 \quad -3.125]$$

$$+$$

$$[-12.5 \quad 6.25 \quad 75 \quad -37.5 \quad -62.5 \quad 31.25]$$

$$+$$

$$[-6.25 \quad -15.625 \quad -62.5 \quad 43.75 \quad 68.75 \quad -28.125]$$

$$= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

column (1, 3, 5) : col (1) + col (3) + col (5)

$$\begin{bmatrix} 18.75 \\ 9.375 \\ -12.5 \\ -6.25 \\ -6.25 \\ -3.125 \end{bmatrix} + \begin{bmatrix} -12.5 \\ 6.25 \\ 75 \\ -37.5 \\ -62.5 \\ 31.25 \end{bmatrix} + \begin{bmatrix} -6.25 \\ -15.625 \\ -62.5 \\ 43.75 \\ 68.75 \\ -28.125 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Rows (2, 4, 6) : Row (2) + Row (4) + Row (6)

$= [9.375 \quad 18.75 \quad 6.25 \quad 12.5 \quad -15.625 \quad -31.25]$

$+$

$[-6.25 \quad 12.5 \quad -37.5 \quad 75 \quad 43.75 \quad -87.5]$

$+$

$[-3.125 \quad -31.25 \quad 31.25 \quad -87.5 \quad -28.125 \quad 118.75]$

$= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$

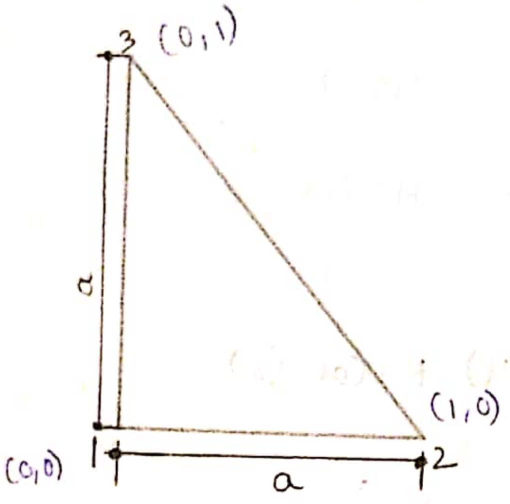
Col (2, 4, 6) : Col (2) + Col. (4) + Col. (6)

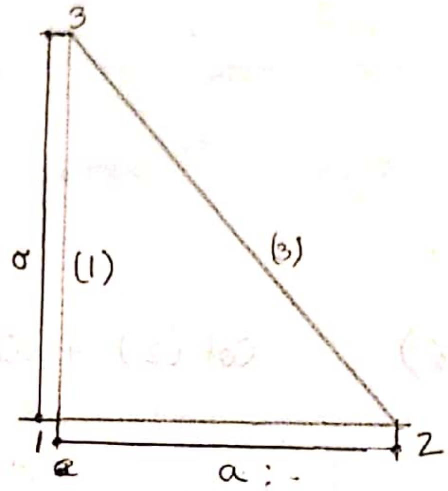
$$= \begin{bmatrix} 9.375 \\ 18.75 \\ 6.25 \\ 12.5 \\ -15.625 \\ -31.25 \end{bmatrix} + \begin{bmatrix} -6.25 \\ 12.5 \\ -37.5 \\ 75 \\ 43.75 \\ -87.5 \end{bmatrix} + \begin{bmatrix} -3.125 \\ -31.25 \\ 31.25 \\ -87.5 \\ -28.125 \\ 118.75 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

<u>Explanation</u>:

- According to Maxwell's Reciprocity theorem, $[K] = [K]^T$ or $K_{ij} = K_{ji}$ [All stiffness matrices are symmetric.]
- In other words, deflection d_i (at coordinate i) due to unit force p_j (at j coordinate) is equal to d_j (at coordinate j) due to p_i (at coordinate j).
- Now, if we add first element of column (1, 3, 5), it means $(K_{11} + K_{13} + K_{15})$, sum of displacements at coordinate 1 due to unit forces p_1 (at coord. 1), p_3 (at coord. 3) and p_5 (at coord. 5) is 0. Similarly, displacements at coordinates
- ~~This signifies~~ 2, 3, 4, 5, 6 is zero.
- This signifies there is a state of <u>equilibrium</u> as no displacements and forces are there.

Assignment - 3.2 (Comp. Structural Mechanics & Dynamics)





For plane triangular domain, $h = a = 1$

$\nu = 0$ (given)

$$K^{(e)} = B^T E B \int_{\Omega^{(e)}} h \, d\Omega$$

$$= \frac{1}{4A} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$x_1 = 0 \quad y_1 = 0 \; ; \; x_2 = 1 \quad y_2 = 0 \; ; \; x_3 = 0 \quad y_3 = 1$

$y_{23} = -1 \quad x_{32} = -1 \; ; \; y_{31} = 1 \quad x_{13} = 0 \; ; \; y_{12} = 0 \quad x_{21} = 1$

$$B = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad E = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} = \frac{E}{1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$= \frac{E}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \frac{1}{2} \det \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{2} = 0.5$$

$$B^T = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$k^{(e)} = \frac{1}{4\times 0.5}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \frac{E}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \frac{E}{4}\begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & -1 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \frac{E}{4}\begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & 2 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

For Bar elements

Stiffness matrix $k^{(e)} = \dfrac{E^{(e)}A^{(e)}}{L^{(e)}}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$

For Element (1), $3 \to 1$, $\alpha_1 = \pi/2$, $A_1 = A$, $L^{(1)} = 1$

$$k^{(1)} = \frac{EA}{1}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = E \begin{array}{c} \begin{array}{cc} 3 & 1 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A & 0 & -A \\ 0 & 0 & 0 & 0 \\ 0 & -A & 0 & A \end{bmatrix} \end{array} \begin{array}{c} 3 \\ \\ 1 \end{array}$$

Element (2), $1 \to 2$, $\alpha_2 = 0$, $A_2 = A$, $L^{(2)} = 1$

$$k^{(2)} = EA\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = E \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} A & 0 & -A & 0 \\ 0 & 0 & 0 & 0 \\ -A & 0 & A & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ \\ 2 \end{array}$$

Element (3), $2 \to 0$, $\alpha_3 = \frac{\pi}{2} + \frac{\pi}{4}$, $A_3 = A'$, $L^{(3)} = \sqrt{2}$

$$k^{(3)} = \frac{EA'}{\sqrt{2}}\begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix} = \frac{EA'}{2\sqrt{2}}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$K_{bar} = K^{(1)} + K^{(2)} + K^{(3)}$

$$= \begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & -A \\ -A & 0 & A+\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} \\ 0 & 0 & -\frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} \\ 0 & 0 & -\frac{A'}{2\sqrt{2}} & +\frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} \\ 0 & -A & +\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}}+A \end{bmatrix}$$

(b) It is impossible to make $K_{bar} = K_{triangle}$ ~~because~~ If ~~you~~ we compare some elements of both the stiffness matrix ~~the~~ such as, K_{12}, K_{14}, K_{15} and more, we found that these elements have zero value for ~~truss~~ ~~and~~ bar and ~~the~~ non-zero for triangle.

However, the diagnol elements can be made equal when $A = \frac{3}{4}$ and $A' = \frac{1}{\sqrt{2}}$

(c) Bar element and triangular elements are two different cases. Bar elements account for ~~nodal~~ forces and displacement at node. While for triangular element forces can be applied at nodes as well as sides. Further, triangle element ~~can~~ have stiffness is more directions as compared to bar element which ~~takes~~ has stiffness in one direction.

So, $K_{bar} \neq K_{triangle}$

(d) When $\nu \neq 0$, $E = \frac{1}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$

$$K^{(e)} = \frac{1}{4A}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \frac{E}{2(1-\nu^2)}\begin{bmatrix} -1 & -\nu & -(1-\nu)/2 \\ -\nu & -1 & -(1-\nu)/2 \\ 1 & \nu & 0 \\ 0 & 0 & (1-\nu)/2 \\ 0 & 0 & (1-\nu)/2 \\ \nu & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \frac{E}{2(1-\nu^2)}\begin{bmatrix} (3-\nu)/2 & (1+\nu)/2 & -1 & -(1-\nu)/2 & -(1-\nu)/2 & -\nu \\ (1+\nu)/2 & (3-\nu)/2 & -\nu & -(1-\nu)/2 & -(1-\nu)/2 & -1 \\ -1 & -\nu & 1 & 0 & 0 & \nu \\ -(1-\nu)/2 & -(1-\nu)/2 & 0 & (1-\nu)/2 & (1-\nu)/2 & 0 \\ -(1-\nu)/2 & -(1-\nu)/2 & 0 & (1-\nu)/2 & (1-\nu)/2 & 0 \\ -\nu & -1 & \nu & 0 & 0 & 1 \end{bmatrix}$$

Poisson ratio is a measure of lateral deformation and longitudional deformation. If $\nu = 0$, then there is no strain in lateral direction due to longitudional stress. It means no lateral stiffness when $\nu = 0$.

~~As ν~~ In case ($\nu \neq 0$), transverse stiffness is there. So, stiffness matrix are different.