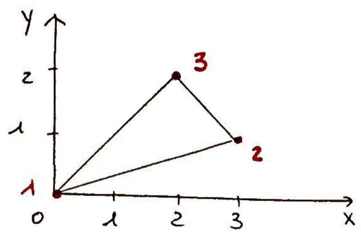
SILVIA PIVETTA (ERASMUS)

COMPUTATIONAL STRUCTURAL MECHANICS and DYNAMICHS

ASSIGMENT 3

ASSIGMENT 3.1

* Compute the entries of Ke for the following plane stress triangle.



$$1.\begin{cases} x_1 = 0 \\ y_1 = 0 \end{cases} \quad 2.\begin{cases} x_2 = 3 \\ y_2 = 1 \end{cases} \quad 3.\begin{cases} x_3 = 2 \\ y_3 = 2 \end{cases}$$

$$E = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix} \qquad h = 1$$

Partial result: $K_{11} = 18.75$
$K_{66} = 118,75$

We have to use the coordinates (area coordinates) to write the shape functions:

$$N_1 = \xi_1 \ ; \ N_2 = \xi_2 \ ; \ N_3 = \xi_3 \quad \rightarrow \begin{cases} x_{ij} = x_i - x_j \\ y_{ij} = y_i - y_j \end{cases}$$

So we have:

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2 y_3 - x_3 y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3 y_1 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1 y_2 - x_2 y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

Now we can write the displacement by:

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \xi_1 & 0 & \xi_2 & 0 & \xi_3 & 0 \\ 0 & \xi_1 & 0 & \xi_2 & 0 & \xi_3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \mathbb{N}$$

We can describe $\mathbb{K}^e$ by the following integral:

$$\mathbb{K}^e = \int_{\Omega^e} h B^T E B \, d\Omega$$

in our problem, it becomes:

$$\mathbb{K}^e = \frac{1}{4A^2} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{12} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \int_{\Omega^e} h \, d\Omega$$

And we know:

$$\begin{cases} y_{21} = -1 \\ x_{21} = 3 \\ y_{23} = -1 \\ y_{31} = 2 \\ x_{32} = -1 \\ x_{13} = -2 \end{cases} \qquad A = \frac{\sqrt{8 \cdot 2}}{2} = 2 \qquad h = 1 \qquad E = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}$$

$$\underset{6\times 3}{\mathbb{K}^e} = \underbrace{\frac{1}{16}}_{\frac{1}{4A^2}} \underbrace{\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \\ -1 & 0 & 3 \\ 0 & 3 & -1 \end{bmatrix}}_{B^T} \underbrace{25 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix}}_{B} \underbrace{\int_{\Omega^e} 1 \, d\Omega =}_{\int_{\Omega^e} h \, d\Omega}$$

So:

$$\mathbb{K}^e = \frac{25}{8} \begin{bmatrix} 6 & 3 & -4 & -2 & -2 & -1 \\ 3 & 6 & 2 & 4 & -5 & -10 \\ -4 & 2 & 24 & -12 & -20 & 10 \\ -2 & 4 & -12 & 24 & 14 & -28 \\ -2 & -5 & -20 & 14 & 22 & -9 \\ -1 & -10 & 10 & -28 & -9 & 38 \end{bmatrix}$$

We can check that $K_{11} = 18{,}75$ and $K_{66} = \cancel{475}\ 118{,}75$.

$$K_{11} = \frac{25 \cdot 6}{8} = \frac{75}{4} = 18{,}75$$

$$K_{66} = \frac{38}{8} \cdot 25 = \frac{475}{4} = 118{,}75.$$

* Show that the sum of the rows (and columns) 1,3,5 of $\mathbb{K}^e$ as well as the sums (and columns) 2,4,6 must vanish, and explain why.

$$\mathbb{K}^e = \frac{25}{8} \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{cccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline 6 & 3 & -4 & -2 & -2 & -1 & 1 \\ 3 & 6 & 2 & 4 & -5 & -10 & 2 \\ -4 & 2 & 24 & -12 & -20 & +10 & 3 \\ -2 & 4 & -12 & 24 & 14 & -28 & 4 \\ -2 & -5 & -20 & 14 & 22 & -9 & 5 \\ -1 & -10 & 10 & -28 & -9 & 38 & 6 \end{array}$$

Sum of the coefficients of the $1^{st}, 3^{rd}, 5^{th}$ rows vectors:

$$
\begin{array}{rl}
[18.75 \quad 9.375 \quad -12.5 \quad -6.25 \quad -6.25 \quad -3.125] & 1^{st} \text{ row} \\
+ [-12.5 \quad 6.25 \quad 75 \quad -37.5 \quad -62.5 \quad 31.25] & 3^{rd} \text{ row} \\
+ [-6.25 \quad -15.625 \quad -62.5 \quad -43.75 \quad -68.75 \quad -28.125] & 5^{th} \text{ row} \\
= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0] &
\end{array}
$$

← the sum of the rows is a zero row.

Sum of columns $1^{st}, 3^{dr}, 5^{th}$:

$$
\underbrace{\frac{25}{8}\begin{bmatrix} 6 \\ 3 \\ -4 \\ -2 \\ -2 \\ -1 \end{bmatrix}}_{1^{st}} + \underbrace{\frac{25}{8}\begin{bmatrix} -4 \\ 2 \\ 24 \\ -12 \\ -20 \\ +10 \end{bmatrix}}_{3^{rd}} + \underbrace{\frac{25}{8}\begin{bmatrix} -2 \\ -5 \\ -20 \\ 14 \\ 22 \\ -9 \end{bmatrix}}_{5^{th}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Sum of the rows $2^{nd}, 4^{th}, 6^{th}$

$$
\begin{array}{rl}
\frac{25}{8}[3 \quad 6 \quad 2 \quad 4 \quad -5 \quad -10] & \cancel{4}\ 2^{nd} \\
+ \frac{25}{8}[-2 \quad 4 \quad -12 \quad 24 \quad 14 \quad -28] & 4^{th} \\
+ \frac{25}{8}[-1 \quad -10 \quad 10 \quad -28 \quad -9 \quad 38] & 6^{th} \\
= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0] &
\end{array}
$$

Sum of the columns $2^{nd}, 4^{th}, 6^{th}$:

$$
\underbrace{\frac{25}{8}\begin{bmatrix} 3 \\ 6 \\ 2 \\ 4 \\ -5 \\ -10 \end{bmatrix}}_{2^{nd}} + \underbrace{\frac{25}{8}\begin{bmatrix} -2 \\ 4 \\ -12 \\ 24 \\ 14 \\ -28 \end{bmatrix}}_{4^{th}} + \underbrace{\frac{25}{8}\begin{bmatrix} -1 \\ -10 \\ 10 \\ -28 \\ -9 \\ 38 \end{bmatrix}}_{6^{th}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

In order to ensure equilibrium, the global stiffness matrix must have zero values for the sum of columns and rows. This is because if it were not, ~~th~~ we would have displacements and forces which would not guarantee the balance of the structure. (equilibrium)

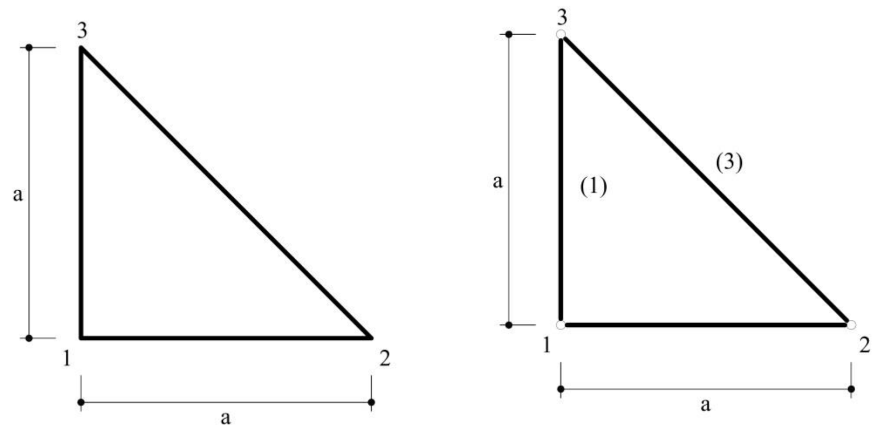
~~In order for~~. According to this we must scruple each column and row as a linear combination which guarantees a linear dependence.

Assignment 3.2

Consider a plane triangular domain of thickness h, with horizontal and vertical edges have length a. Let's consider for simplicity a = h = 1. The material parameters are E,ν. Initially ν is set to zero. Two structural models are considered for this problem as depicted in the figure:

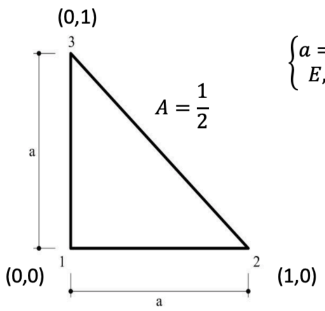
♣ A plane linear Turner triangle with the same dimensions.
♣ A set of three bar elements placed over the edges of the triangular domain. The cross

sections for the bars are $A_1 = A_2$ and A_3.



a) ***Calculate the stiffness matrix K^e for both models.***

- **Turner triangle:**



$$\begin{cases} a = h = 1 \\ E, \upsilon = 0 \end{cases}$$

We can define the element stiffness matrix as:

$$K^e = \int_{\Omega^e} hB^T EB d\Omega$$

We assume the thickness constant, so according to this we can write the element stiffness matrix as:

$$\mathbf{K}^e = \frac{1}{4A} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

where the coordinates of each nodes are $\begin{cases} x_{jk} = x_j - x_k \\ y_{jk} = y_j - y_k \end{cases}$

Considering the plane stress case with υ = 0, the constitutive matrix it becomes:

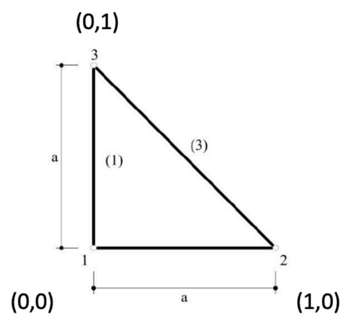
$$E = \begin{bmatrix} E_{11} & E_{21} & E_{31} \\ E_{12} & E_{22} & E_{32} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}$$

$$E = E\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Substituting all the nodes coordinates values into the matrix:

$$\mathbf{K}_{\mathbf{triangle}} = \frac{E}{4}\begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ & 3 & 0 & -1 & -1 & -2 \\ & & 2 & 0 & 0 & 0 \\ & & & 1 & 1 & 0 \\ & & & & 1 & 0 \\ Symm. & & & & & 2 \end{bmatrix}$$

- **Tree bar elements (υ = 0):**



We can define the stiffness K^e matrix of each bar elements as:

$$K^e = \frac{EA^e}{L^{(e)}}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

Where $\begin{cases} s = \sin\alpha \\ c = cos\alpha \end{cases}$ and $\alpha = angle$ between the local coordinate system of the bar with respect to the global coordinate system.

1. Element (1) $\alpha = 90°$

$$K^{(1)} = \frac{EA_1}{a}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

2. Element (2) $\alpha = 0°$

$$K^{(2)} = \frac{EA_2}{a}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3. Element (3) $\alpha = 135°$

$$K^{(3)} = \frac{EA_3}{a\sqrt{2}}\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

According to the fact that $A_1 = A_2 = A\ and\ A_3 = A'$ the global stiffness matrix becomes:

$$\mathbf{K_{bar}} = E\begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ & A & 0 & 0 & 0 & -A \\ & & A + A'/2\sqrt{2} & -A'/2\sqrt{2} & -A'/2\sqrt{2} & A'/2\sqrt{2} \\ & & & A'/2\sqrt{2} & A'/2\sqrt{2} & -A'/2\sqrt{2} \\ & & & & A'/2\sqrt{2} & -A'/2\sqrt{2} \\ Symm. & & & & & A + A'/2\sqrt{2} \end{bmatrix}$$

b) ***Is there any set of values for cross sections A1=A2=A and A3=A' to make both stiffness matrix equivalent: Kbar = Ktriangle? If not, which are these values to make them as similar as possible?***

It is impossible to make off-diagonals elements match; that's is evident because some k_{ij} elements in the bar are equal to zero, and the same k_{ij} elements In the triangle are $\neq 0$. We have to check what are the values that are $k_{ij} \neq 0$ in both case.

$$\begin{bmatrix} k_{11} & 0 & k_{13} & 0 & 0 & 0 \\ 0 & k_{22} & 0 & 0 & 0 & k_{26} \\ k_{31} & 0 & k_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{44} & k_{45} & 0 \\ 0 & 0 & 0 & k_{54} & k_{55} & 0 \\ 0 & k_{62} & 0 & 0 & 0 & k_{66} \end{bmatrix}$$

1. $A_1 = A_2 = A$

   We have to look the elements that are $\neq 0$, whitch are:

$$k_{11}, k_{22} \rightarrow A = 0{,}75$$

$$k_{13}, k_{26} \rightarrow A = 0{,}5$$

Now for those values we want to obtain the most similar numbers in all the k_{ij} elements involved, so we can use:

$$A = \frac{0{,}5 + 0{,}75}{2} = 0{,}625$$

2. $A_3 = A$

$$k_{33} \rightarrow A + \frac{A'}{2\sqrt{2}} = 0{,}5 \rightarrow A' = -\frac{\sqrt{2}}{2} \rightarrow this\ solutions\ is\ not\ possible$$

$$k_{44}, k_{45}, k_{55} \rightarrow \frac{A'}{2\sqrt{2}} = 0{,}25 \rightarrow A' = \frac{\sqrt{2}}{2}$$

$$k_{66} \rightarrow \frac{A'}{2\sqrt{2}} = 0{,}5 \rightarrow A' = \sqrt{2}$$

We are going to use the second value that we obtained because with value we are able to obtain four equal values in the matrices. According to this the other two remaining elements will not be identical to each other, but will not differ much either.

c) ***Why these two stiffness matrix are not equivalent? Find a physical explanation.***

We know that the stiffness matrix is a relationship between displacements and forces in the nodes.

The triangle supports both axial forces along its length and directional loads. In order for this new adjunctive condition to be satisfied, we must introduce form functions.

If we instead analyze the 2d bar element, which is a simple 2-node finite element, it resists an internal axial force only along its longitudinal dimension (in fact if we consider the force it can be shared by the nodes 1 and 2 but not from node 3).

in fact, the bar elements transmit only axial forces and the ability to share the force depends on the connection of the nodes themselves; the nodes in the triangle, since they are connected, they resist forces in a combined way.

d) Solve question considering ν≠0 and extract some conclusions.

if we consider, the stiffness matrix will remain unchanged. Therefore the stiffness matrix of the triangle will be:

$$\mathbf{K}_{\textbf{triangle}} = \frac{E}{4(1-\nu^2)} \begin{bmatrix} 3-\nu & 1+\nu & -2 & \nu-1 & \nu-1 & -2\nu \\ & 3-\nu & -2\nu & \nu-1 & \nu-1 & -2 \\ & & 2 & 0 & 0 & 2\nu \\ & & & 1-\nu & 1-\nu & 0 \\ & & & & 1-\nu & 0 \\ Sym. & & & & & 2 \end{bmatrix}$$