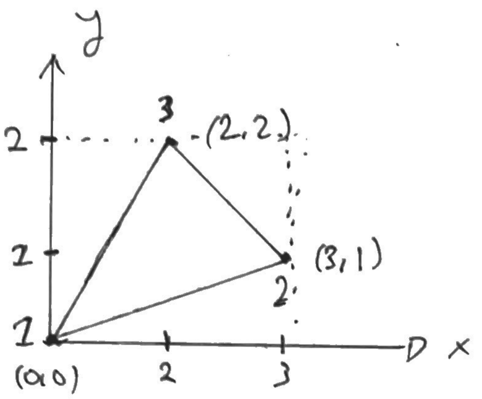
Assignment 3.1

1)



$$E = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}$$

$h = 1 \Rightarrow$ constant

$x_{ij} = x_i - x_j$

$y_{ij} = y_i - y_j$

$$A = 3 \cdot 2 - \frac{3 \cdot 1}{2} - 2 \cdot 2/2 - 1 \cdot 1/2 = 2$$

For constant thickness:

$$\mathbb{K}^e = \frac{h}{4A} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$y_{23} = y_2 - y_3 = 1 - 2 = -1$

$y_{31} = y_3 - y_1 = 2 - 0 = 2$

$y_{12} = y_1 - y_2 = 0 - 1 = -1$

$x_{32} = x_3 - x_2 = 2 - 3 = -1$

$x_{13} = x_1 - x_3 = 0 - 2 = -2$

$x_{21} = x_2 - x_1 = 3 - 0 = 3$

$$\Rightarrow \mathbb{K}^e = \frac{1}{4\cdot 2}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \\ -1 & 0 & 3 \\ 0 & 3 & -1 \end{bmatrix}\begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -100 & -25 & -50 \\ -25 & -100 & -50 \\ 200 & 50 & -100 \\ -50 & -200 & 100 \\ -100 & -25 & 150 \\ 75 & 300 & -50 \end{bmatrix}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 150 & 75 & -100 & -50 & -50 & -25 \\ 75 & 150 & 50 & 100 & -125 & -250 \\ -100 & 50 & 200 & -300 & -500 & 250 \\ -50 & 100 & -300 & 600 & 350 & -700 \\ -50 & -125 & -500 & 350 & 550 & -225 \\ -25 & -250 & 250 & -700 & -225 & 950 \end{bmatrix}$$

$$= \begin{bmatrix} 18{,}75 & 9{,}375 & -12{,}5 & -6.25 & -6{,}25 & -3.125 \\ 9{,}375 & 18{,}75 & 6.25 & 12{,}5 & -15{,}625 & -31{,}25 \\ -12{,}5 & 6{,}25 & 75 & -37{,}5 & -62{,}5 & 31.25 \\ -6{,}25 & 12{,}5 & -37{,}5 & 75 & 43{,}75 & -87{,}5 \\ -6{,}25 & -15{,}625 & -62{,}5 & 43{,}75 & 68{,}75 & -28{,}125 \\ -3{,}125 & -31{,}25 & 31.25 & -87{,}5 & -28{,}125 & 118{,}75 \end{bmatrix}$$

2) Row 1 + Row 3 + Row 5 = Column 1 + Column 3 + Column 5

$= 18{,}75 + 9{,}375 + -12{,}5 - 6{,}25 - 6{,}25 - 3{,}125$

$-12{,}5 + 6{,}25 + 75 - 37{,}5 - 62{,}5 + 31{,}25$

$-6{,}25 - 15{,}625 - 62{,}5 + 43{,}75 + 68{,}75 - 28{,}125$

$\underline{\underline{= 0}}$

Row 2 + Row 4 + Row 6

= Column 2 + Column 4 + Column 6

$= 9{,}375 + 18{,}75 + 6{,}25 + 12{,}5 + 15{,}625 - 31{,}25$

$-6{,}25 + 12{,}5 - 37{,}5 + 75 + 43 - 87{,}5$

$-3{,}125 - 31{,}25 + 31{,}25 - 87{,}5 - 28{,}125 + 118{,}75$

$\underline{\underline{= 0}}$

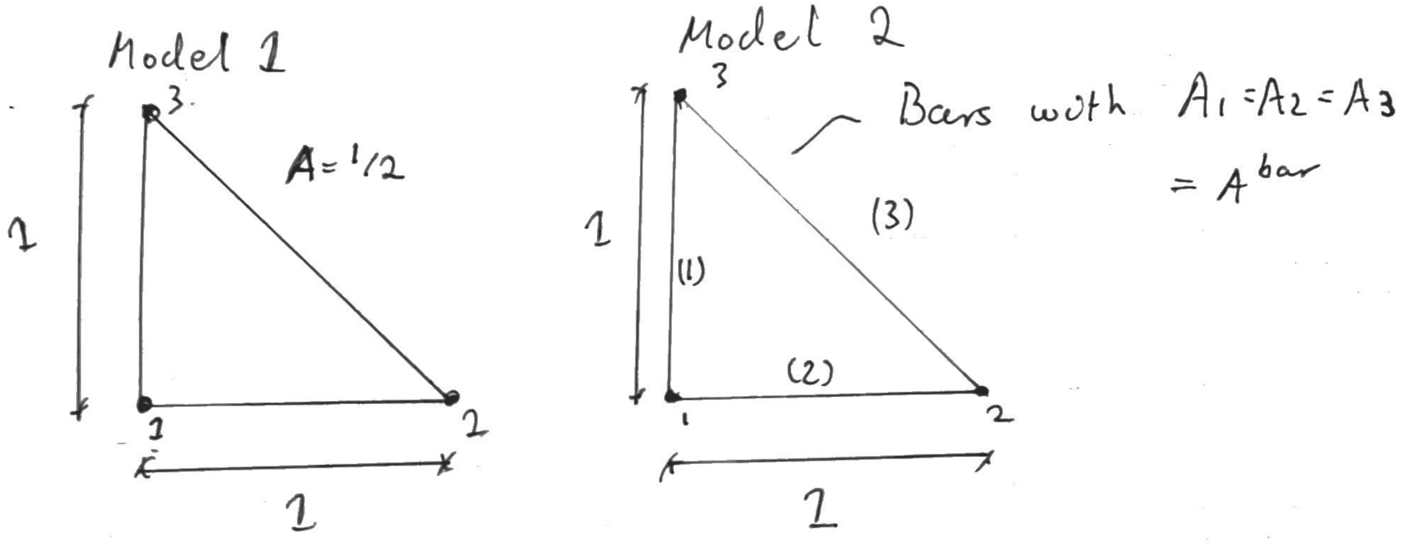
The rows and columns are linearly dependent

$\Rightarrow$ $\mathbb{K}^e$ is singular

$\Rightarrow$ Makes sense since we dont have any BC's.

Assignment 3.2

Two structural models:



a) For now $\nu = 0$

Model 1

Repeating what was done in assignment 3.1

with :

$$y_{23} = y_2 - y_3 = -1 \qquad x_{32} = x_3 - x_2 = -1$$

$$y_{31} = y_3 - y_1 = 1 \quad \text{and} \quad x_{13} = x_1 - x_3 = 0$$

$$y_{12} = y_1 - y_2 = 0 \qquad x_{21} = x_2 - x_1 = 1$$

and

$$\mathbb{E} = E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

↳

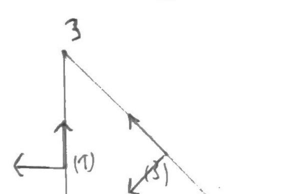
We get:

$$\mathbb{K}^{triangle} = \frac{1}{4\ \frac{1}{2}} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\overset{\text{Matlab}}{=} E \begin{bmatrix} 0,75 & 0,25 & -0,5 & -0,25 & -0,25 & 0 \\ 0,25 & 0,75 & 0 & -0,25 & -0,25 & -0,5 \\ -0,5 & 0 & 0,5 & 0 & 0 & 0 \\ -0,25 & -0,25 & 0 & 0,25 & 0,25 & 0 \\ -0,25 & -0,25 & 0 & 0,25 & 0,25 & 0 \\ 0 & -0,5 & 0 & 0 & 0 & 0,5 \end{bmatrix}$$

<u>Model 2</u> Using direct stiffness method

$$K^e = \frac{E^e A^e}{L^e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix} \qquad \begin{aligned} c &= \cos\theta \\ s &= \sin\theta \end{aligned}$$



Element 1 : $\Theta = \pi/2$, $E^{(1)} = E$, $A^{(1)} = A^{bar}$, $L^{(1)} = 1$

$$\Rightarrow \mathbb{K}^{(1)} \frac{EA^{(1)}}{1} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

- Connected to node 1 and 3

Element 2 : $\Theta = 0$, $E^{(2)} = E$, $A^{(2)} = A^{bar}$, $L^{(2)} = 1$

$$\mathbb{K}^{(2)} = \frac{EA^{(2)}}{1} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Connected to node 1 and 2

Element 3 : $\Theta = 3\pi/4$, $E^{(3)} = E$, $A^{(3)} = A^{bar}$, $L^{(3)} = \sqrt{2}$

$$\mathbb{K}^{3} = \frac{EA^{(3)}}{2 \cdot \sqrt{2}} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

- Connected to node 2 and 3

$$\mathbb{K}^{bar} = \frac{E}{\sqrt{2} \cdot 2} \begin{bmatrix} 2\sqrt{2}A^{(2)} & 0 & -2\sqrt{2}A^{(2)} & 0 & 0 & 0 \\ 0 & 2\sqrt{2}A^{(1)} & 0 & 0 & 0 & -2\sqrt{2}A^{(1)} \\ -2\sqrt{2}A^{(2)} & 0 & 2\sqrt{2}A^{(2)} + A^{(3)} & -A^{(3)} & -A^{(3)} & A^{(3)} \\ 0 & 0 & -A^{(3)} & A^{(3)} & -A^{(3)} & -A^{(3)} \\ 0 & 0 & -A^{(3)} & -A^{(3)} & A^{(3)} & -A^{(3)} \\ 0 & -2\sqrt{2}A^{(1)} & A^{(3)} & -A^{(3)} & -A^{(3)} & 2\sqrt{2}A^{(1)} + A^{3} \end{bmatrix}$$

b)

Now $A^{(1)} = A^{(2)} = A$ and $A^{(3)} = A'$

It is not possible to chose A and A' such that $\mathbb{K}^{triangle} = \mathbb{K}^{bar}$, but we obtain the most similar matrices if

$A^{(1)} = A^{(2)} \approx 0{,}25$ and $A^{(3)} = A' = \sqrt{2}/2$

This way the last 4 diagonals are correct. $K_{33}, K_{44}, K_{55}, K_{66}$

c)

The two stiffness matrices are not equal because the two structures are different. Model 1 is not considered hollow in the midle while model 2 is.

Model 2 only have stiffness in the respective bars directions, while model 1 has stiffness in all directions.

d)

Model 1

Now :

$$\mathbb{K}^{triangle} = \frac{1}{2}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} E/_{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Matlab.

$$= \frac{E}{2(1-\nu^2)}\begin{bmatrix} (3/2-\nu/2) & (\nu/2+1/2) & -1 & (\nu/2-1/2) & (\nu/2-1/2) & -\nu \\ (\nu/2+1/2) & (3/2-\nu/2) & -\nu & (\nu/2-1/2) & (\nu/2-1/2) & -1 \\ -1 & -\nu & 1 & 0 & 0 & \nu \\ (\nu/2-1/2) & (\nu/2-1/2) & 0 & (1/2-\nu/2) & (1/2-\nu/2) & 0 \\ (\nu/2-1/2) & (\nu/2-1/2) & 0 & (1/2-\nu/2) & (1/2-\nu/2) & 0 \\ -\nu & -1 & \nu & 0 & 0 & 1 \end{bmatrix}$$

Model 2

$\mathbb{K}^{bar}$ will be equal.

$\Rightarrow$ When $\nu \neq 0$ the stiffness matrix becomes more stiff.

This is because we consider the material to also obtain strains in the transversal directions when $\nu \neq 0$.