Computational Structural Mechanics and Dynamics

As3 FEM Plane Stress Problem and Linear Triangle

Ye Mao

mao.ye@estudiant.upc.edu

Master of Numerical methods on engineering - Universitat Politècnica de Catalunya

Assignment 3.1

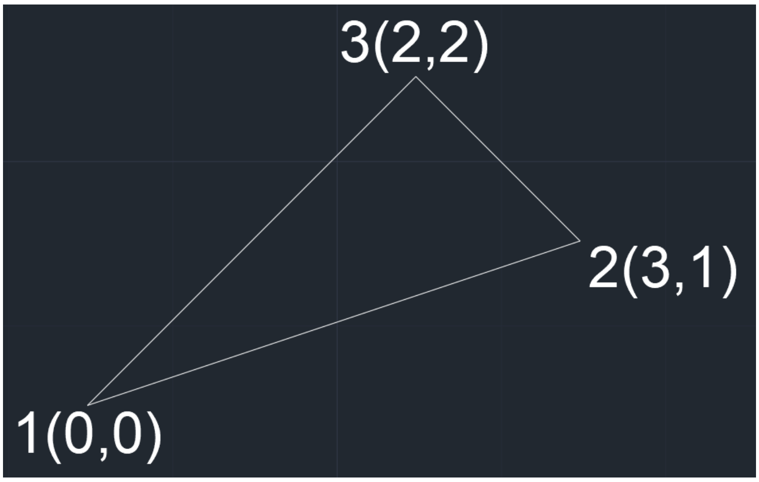
1. **Compute the entries of K_e for the following plane stress triangle:**

$$x_1 = 0, y_1 = 0, x_2 = 3, y_2 = 1, x_3 = 2, y_3 = 2,$$

$$E = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}, h = 1$$

Partial result: $K_{11} = 18.75, K_{66} = 118.75.$

[Answer]



$$K^e = \frac{h}{4A} B^T E B$$

$$= \frac{h}{4A} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

Where $h = 1, A = 2.$

$$K^e = \frac{1}{4 \cdot 2} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \\ -1 & 0 & 3 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix}$$

$$K^e = \begin{bmatrix} 18.75 & 9.375 & -12.5 & -6.25 & -6.25 & -3.125 \\ 9.375 & 18.75 & 6.25 & 12.5 & -15.625 & -31.25 \\ -12.5 & 6.25 & 75 & -37.5 & -62.5 & 31.25 \\ -6.25 & 12.5 & -37.5 & 75 & 43.75 & 87.5 \\ -6.25 & -15.625 & -62.5 & 43.75 & 68.75 & -28.125 \\ -3.125 & -31.25 & 31.25 & -87.5 & -28.125 & 118.75 \end{bmatrix}$$

2. **Show that the sum of the rows (and columns)1, 3 and 5 of K^e as well as the sum of rows (and columns) 2, 4 and 6 must vanish, and explain why.**

[Answer]

All the sum of rows (and columns) are equal to 0.

We can assume the displacement is

$$\begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \end{bmatrix}$$

If the sum of rows (and columns)1,3,5 equal to 0, it means no reaction for the structure in x coordinate. Meanwhile, if the sum of rows (and columns)2,4,6 equal to 0, it means no reaction for the structure in y coordinate.

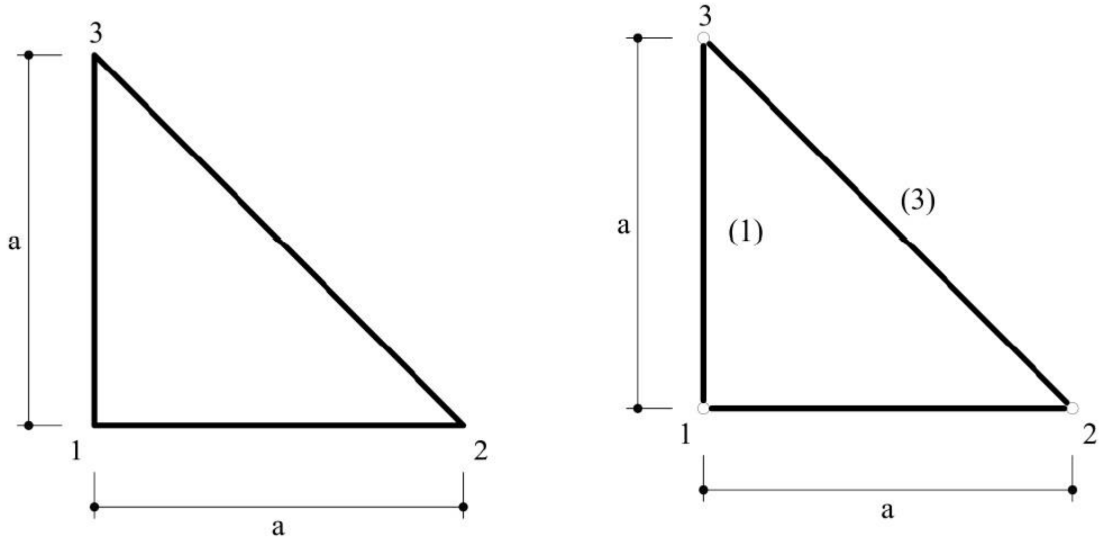
Then, the structure reaches the rigid body motion.

Assignment 3.2

Consider a plane triangular domain of thickness h, with horizontal and vertical edges have length a. Let's consider for simplicity $a = h = 1$. The material parameters are E, ν. Initially ν is set to zero. Two structural models are considered for this problem as depicted in figure:

A plane linear Turner triangle with the same dimensions.

A set of three bar elements placed over the edges of the triangular domain. The cross sections for the bars are $A_1 = A_2$ and A_3.



a) Calculate the stiffness matrix K^e for both models.

[Answer]

Firstly, we calculate the plane linear Turner triangle with the same dimensions.

$$K^e = \frac{h}{4A} B^T E B$$

$$= \frac{h}{4A} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

Substituting the numerical values:

$$K^e = \frac{E}{2(1-\nu^2)} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$K^e = \frac{E}{2(1-\nu^2)}\begin{bmatrix} \frac{3-\nu}{2} & \frac{1+\nu}{2} & -1 & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -\nu \\ \frac{1+\nu}{2} & \frac{3-\nu}{2} & -\nu & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -1 \\ -1 & -\nu & 1 & 0 & 0 & \nu \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 \\ -\nu & -1 & \nu & 0 & 0 & 1 \end{bmatrix}$$

While $\nu = 0$ the stiffness matrix is the following:

$$K^e = \frac{E}{4}\begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Secondly, we calculate the set of three bar elements placed over the edges of the triangular domain.

$$K^1 = EA\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

$$K^2 = EA\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^3 = \frac{EA'}{2\sqrt{2}}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $A^* = \frac{A\prime}{2\sqrt{2}}$, then assembly

$$K = E\begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & -A \\ -A & 0 & A+A^* & A^* & -A^* & -A^* \\ 0 & 0 & A^* & A^* & -A^* & -A^* \\ 0 & 0 & -A^* & -A^* & A^* & A^* \\ 0 & -A & -A^* & -A^* & A^* & A+A^* \end{bmatrix}$$

b) Is there any set of values for cross section $A_1 = A_2 = A$ and $A_3 = A'$ to make both stiffness matrix equivalent： $K_{bar} = K_{triangle}$? If not, which are these values to make them as similar as possible?

[Answer]

The two matrixes have no similar structure at all. Even for the diagonal term is hard to find a value of A and A^* to make the two matrixes similar.

If we let $A = \frac{3}{4}$ and $A + A^* = \frac{2}{4}$, then $A^* = -\frac{1}{4}$.

A^* is negative making no physical sense.

c) **Why these two stiffness matrix are not equivalent? Find a physical explanation.**

[Answer]

The two matrixes present their major difference on the off-diagonal term. Because there is no distortion energy stored in the process of deformation. Only the axial tension and compression produces reaction terms while in the Turner element distortion plays an important role.

d) **Solve question a) considering $\boldsymbol{\nu \neq 0}$ and extra some conclusion.**

Note: To solve this assignment it's recommended to check the features of the linear triangle in presentation " CSMD_05_linear_Triangle ". Some comments will be given in the next class.

[Answer]

Following the solution in question a)

We obtain

$$K^e = \frac{E}{2(1-\nu^2)}\begin{bmatrix} \frac{3-\nu}{2} & \frac{1+\nu}{2} & -1 & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -\nu \\ \frac{1+\nu}{2} & \frac{3-\nu}{2} & -\nu & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -1 \\ -1 & -\nu & 1 & 0 & 0 & \nu \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 \\ -\nu & -1 & \nu & 0 & 0 & 1 \end{bmatrix}$$

When taking into account the effect of Poisson Ratio, the stiffness on the diagonal terms is reduced. The deformation due to the Poisson effect is in the same direction than the imposed by the external forces. To keep the equilibrium, other terms which are the most of off-diagonal terms increase by the Poisson's ratio.