Computational Structural Mechanics and Dynamics
Assignment 3

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1 Problem Description

1.1 Assignment 3.1

1. Compute the entries of K_e for the following plane stress triangle:

$$x_1 = 0, y_1 = 0, x_2 = 3, y_2 = 1, x_3 = 2, y_3 = 2 \tag{1}$$

$$C = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}, \quad h = 1 \tag{2}$$

2. Show that the sum of the rows (and columns) 1, 3 and 5 of K_e as well as the sum of rows (and columns) 2, 4 and 6 must vanish, and explain why.

1.2 Assignment 3.2

Consider a plane triangular domain of thickness h, with horizontal and vertical edges of length a. Let's consider a = h = 1 for simplicity. The material parameters are E, v. Initially v is set to zero. Two structural models are considered for this problem as depicted in the figure:

- A plane linear Turner triangle with the same dimensions.
- A set of three bar elements placed over the edges of the triangular domain. The cross sections for the bars are $A_1 = A_2$ and A_3.

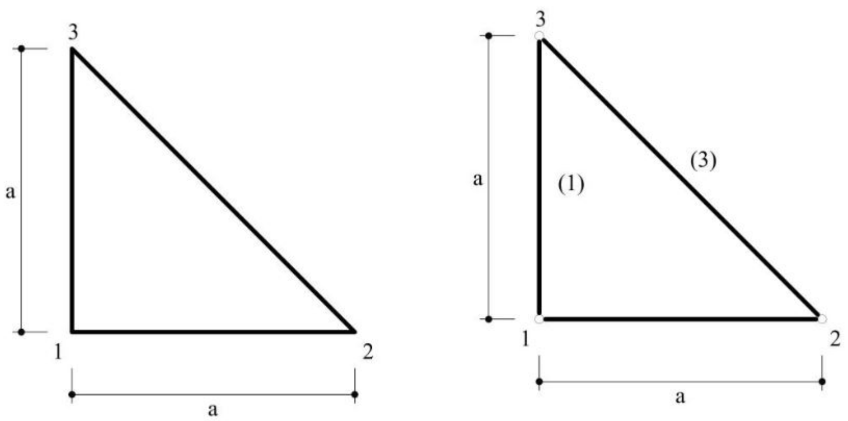

Figure 1: Structural Models - Assignment 3.2

a) Calculate the stiffness matrix K_e for both models.

b) Is there any set of values for cross sections $A_1 = A_2 = A$ and $A_3 = A'$ to make both stiffness matrices equivalent: $K_{bar} = K_{triangle}$? If not, which are these values to make them as similar as possible?

c) Why are these two stiffness matrices not equivalent? Find a physical explanation.

d) Solve question a) considering $v \neq 0$ and extract some conclusions.

2 Solution

2.1 Assignment 3.1

For a 2D triangular element, the stiffness matrix is given by:

$$K = h \int_A B^T C B dA \tag{3}$$

Where:

- h is the thickness of the element
- B is the kinematic matrix, given by:

$$B = \frac{1}{2A} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \tag{4}$$

- A is the area of the element
- $b_i = y_j - y_k$
- $c_i = x_k - x_j$
- C is the constitutive tensor, which in this case is the plane stress tensor given by equation (2).

In order to compute the stiffness matrix, we must first obtain the kinematic matrix of the element:

$$b_1 = -1 \qquad b_2 = 2 \qquad b_3 = -1 \tag{5}$$

$$c_1 = -1 \qquad c_2 = -2 \qquad c_3 = 3 \tag{6}$$

$$2A = 4 \tag{7}$$

$$B = \frac{1}{4} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix} \tag{8}$$

Then, by applying equation (3) we compute the stiffness matrix K:

$$K = \begin{bmatrix} 18.75 & 9.375 & -12.50 & -6.25 & -6.25 & -3.125 \\ & 18.75 & 6.25 & 12.50 & -15.625 & -31.25 \\ & & 75.00 & -37.50 & -62.50 & 31.25 \\ & & & 75.00 & 43.75 & -87.50 \\ & & & & 68.75 & -28.125 \\ & & & & & 118.75 \end{bmatrix} \tag{9}$$

It may be observed that the sum of all matrix elements corresponding to horizontal degrees of freedom (1, 3, and 5) of any given row or column is zero, and the same applies for vertical degrees of freedom (2, 4 and 6). For example, $K_{11} + K_{13} + K_{15} = 0$. The explanation for this comes from the physical meaning of the stiffness matrix.

Element ij of a stiffness matrix represents the force that must be applied to degree of freedom i when degree of freedom j is subjected to a unitary displacement, while the remaining degrees of freedom remain restrained. Therefore, the reason for the sum of these elements going to zero is equilibrium: If degree of freedom 1 is being subjected to a unitary displacement by applying the force corresponding to K_{11} on it, degrees of freedom 3 and 5 must be held in place (restrained) by applying the forces corresponding to K_{31} and K_{51} on them, respectively. This implies that the structure is in equilibrium.

2.2 Assignment 3.2

2.2.1 Plane Linear Triangle Model

The plane stress constitutive tensor as a function of E and v takes the form:

$$C = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ & 1 & 0 \\ & & \frac{1-v}{2} \end{bmatrix} \tag{10}$$

To compute the stiffness matrix for the plane triangle model, the same procedure applied for Assignment 3.1 was implemented, obtaining the following results:

$$b_1 = -1 \qquad b_2 = 1 \qquad b_3 = 0 \tag{11}$$

$$c_1 = -1 \qquad c_2 = 0 \qquad c_3 = 1 \tag{12}$$

$$2A = 1 \tag{13}$$

Replacing these values in equation (4) yields:

$$B = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix} \tag{14}$$

Then, by applying equation (3) and considering that $v = 0$, we compute the stiffness matrix K:

$$K_{triangle} = E\begin{bmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & 0 \\ & \frac{3}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} \\ & & \frac{1}{2} & 0 & 0 & 0 \\ & & & \frac{1}{4} & \frac{1}{4} & 0 \\ & & & & \frac{1}{4} & 0 \\ & & & & & \frac{1}{2} \end{bmatrix} \tag{15}$$

2.2.2 Bar Structure Model

In order to compute the stiffness matrix of the truss model, we must first know the nodal connectivity of the structural elements and compute the elemental stiffness matrices to then assemble the global matrix. For this structural configuration, we have the following connectivity and elemental geometry:
The stiffness matrix for any given element in global coordinates in a 2D truss structure is given by:

$$k^{(e)} = \frac{EA}{l^{(e)}}\begin{bmatrix} \eta^2 & \eta\mu & -\eta^2 & -\eta\mu \\ & \mu^2 & -\eta\mu & -\mu^2 \\ & & \eta^2 & \eta\mu \\ & & & \mu^2 \end{bmatrix} \tag{16}$$

Element	Node 1	Node 2	Length	Cross Section Area	Orientation (rad)
1	1	3	1	A	$\frac{\pi}{2}$
2	1	2	1	A	0
3	2	3	$\sqrt{2}$	A'	$\frac{3\pi}{4}$

Table 1: Node Connectivity and Element Geometry

Where:

$l^{(e)}$ is the length of element e.

$\eta = cos(\theta)$

$\mu = sin(\theta)$

θ is the angle of orientation of the element with respect to the positive global x axis.

Hence, computing the elemental stiffness matrices yields:

$$k^{(1)} = EA \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 1 & 0 & -1 \\ & & 0 & 0 \\ & & & 1 \end{bmatrix} \tag{17}$$

$$k^{(2)} = EA \begin{bmatrix} 1 & 0 & -1 & 0 \\ & 0 & 0 & 0 \\ & & 1 & 0 \\ & & & 0 \end{bmatrix} \tag{18}$$

$$k^{(3)} = EA'\frac{\sqrt{2}}{4} \begin{bmatrix} 1 & -1 & -1 & 1 \\ & 1 & 1 & -1 \\ & & 1 & -1 \\ & & & 1 \end{bmatrix} \tag{19}$$

Taking into account the contribution to each element to the global stiffness matrix according to the node connectivity matrix, we then compute the global stiffness matrix of the structural system:

$$K_{bar} = E \begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ & A & 0 & 0 & 0 & -A \\ & & A + \frac{\sqrt{2}}{4}A' & -\frac{\sqrt{2}}{4}A' & -\frac{\sqrt{2}}{4}A' & \frac{\sqrt{2}}{4}A' \\ & & & \frac{\sqrt{2}}{4}A' & \frac{\sqrt{2}}{4}A' & -\frac{\sqrt{2}}{4}A' \\ & & & & \frac{\sqrt{2}}{4}A' & -\frac{\sqrt{2}}{4}A' \\ & & & & & A + \frac{\sqrt{2}}{4}A' \end{bmatrix} \tag{20}$$

2.2.3 Comparison Between Models and Discussion

- Case 1 ($v = 0$)

Since $K_{triangle}$ and K_{bar} do not share the same non-zero elements, **it is not possible to make them equivalent**. Now, to find the values of A and A' that will make the matrices as similar as possible, we will force equality between elements 1,1 and 5,5 of both matrices, which leads to the following results for the case where v = 0:

$$\frac{3E}{4} = EA \qquad \rightarrow \qquad A = \frac{3}{4} \tag{21}$$

$$\frac{E}{4} = \frac{\sqrt{2}}{4}EA' \qquad \rightarrow \qquad A' = \sqrt{\frac{1}{2}} \tag{22}$$

Elements of the bar structure model can only withstand axial loads, which for example results in element 2 (horizontal element) not contributing any stiffness to vertical degrees of freedom since these are perpendicular to its axis. Meanwhile, this is not the case for the plane triangle model, in which the structure behaves as a continuous 2D elastic body within the domain as long as the loads remain on the mid plane of the structure and are parallel to it.

- Case 2 ($v \neq 0$)

Considering that now $v \neq 0$, the stiffness matrix of the triangle model becomes:

$$K_{triangle} = E \begin{bmatrix} \frac{(v-3)}{4(v^2-1)} & \frac{-1}{4(v-1)} & \frac{1}{2(v^2-1)} & \frac{-1}{4(v+1)} & \frac{-1}{4(v+1)} & \frac{v}{2(v^2-1)} \\ & \frac{(v-3)}{4(v^2-1)} & \frac{v}{2(v^2-1)} & \frac{-1}{4(v+1)} & \frac{-1}{4(v+1)} & \frac{1}{2(v^2-1)} \\ & & \frac{-1}{2(v^2-1)} & 0 & 0 & \frac{-v}{2(v^2-1)} \\ & & & \frac{1}{4(v+1)} & \frac{1}{4(v+1)} & 0 \\ & & & & \frac{1}{4(v+1)} & 0 \\ & & & & & \frac{-1}{2(v^2-1)} \end{bmatrix} \tag{23}$$

By introducing a Poisson coefficient in the constitutive tensor, the behavior of the structure is modified since the application of horizontal stresses will result in horizontal deformations and vice versa. It may be observed that all elements of the matrix depend on both the Young's modulus and the Poisson ratio of the material.

3 References

- Hurtado Gómez, J.E. Análisis Matricial de Estructuras. Universidad Nacional de Colombia.