Universitat Politècnica de Catalunya

Escola Tècnica Superior d'Enginyers de Camins, Canals i Ports

MASTER EN INGENIERÍA ESTRUCTURAL Y DE LA CONSTRUCCIÓN

Asignatura:

COMPUTATIONAL STRUCTURAL MECHANICS AND DYNAMICS

Assignment 3

On "The Plane Stress Problem"

And "The 3-node Plane Stress Triangle"

By

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Assignment 3.1:

On "The Plane Stress Problem":

In isotropic elastic materials (as well as in plasticity and viscoelasticity) it is convenient to use the so-called Lamé constants λ and μ instead of E and ν in the constitutive equations. Both λ and μ have the physical dimension of stress and are related to E and ν

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}; \mu = G = \frac{E}{2(1+\nu)}$$

1. Find the inverse relations for E,ν in terms of λ,μ.
2. Express the elastic matrix for plane stress and plane strain cases in terms of λ, μ.
3. Split the stress-strain matrix E for plane strain

$$\boldsymbol{E} = \boldsymbol{E}_{\lambda} + \boldsymbol{E}_{\mu}$$

in which $\boldsymbol{E}_{\mu}$ and $\boldsymbol{E}_{\lambda}$contain only μ and λ, respectively.

This is the Lamé {λ,μ} splitting of the plane strain constitutive equations, which leads to the so-called B-bar formulation of near-incompressible finite elements.

4. Express $\boldsymbol{E}_{\lambda}$ and $\boldsymbol{E}_{\mu}$ also in terms of E and ν.

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1. Operating

$$\mu = \frac{E}{2(1+\nu)} \rightarrow E = 2\mu(1+\nu)$$

Substituting

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{2\mu(1+\nu)\nu}{(1+\nu)(1-2\nu)} = \frac{2\mu\nu}{(1-2\nu)}$$

$$1-2\nu = \frac{2\mu}{\lambda}\nu \rightarrow 1 = 2\nu\left(\frac{\mu}{\lambda}+1\right) = 2\nu\frac{(\mu+\lambda)}{\lambda}$$

$$\nu = \frac{\lambda}{2(\mu+\lambda)}$$

Backing to first equation

$$E = 2\mu(1+\nu) = 2\mu\left(1+\frac{\lambda}{2(\mu+\lambda)}\right) = 2\mu\left[\frac{(2\mu+2\lambda+\lambda)}{2(\mu+\lambda)}\right] = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda}$$

2. For isotropic material characterized by Young modulus (E) and Poison coefficient (ν) the elastic matrix for plane stress is

$$\boldsymbol{E} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Substituting

$$\boldsymbol{E}=\frac{\dfrac{\mu(2\mu+3\lambda)}{\mu+\lambda}}{1-\left(\dfrac{\lambda}{2(\mu+\lambda)}\right)^2}\begin{bmatrix}1 & \dfrac{\lambda}{2(\mu+\lambda)} & 0\\ \dfrac{\lambda}{2(\mu+\lambda)} & 1 & 0\\ 0 & 0 & \dfrac{1-\dfrac{\lambda}{2(\mu+\lambda)}}{2}\end{bmatrix}$$

Simplifying

$$\boldsymbol{E}=\frac{\dfrac{\mu(2\mu+3\lambda)}{\mu+\lambda}}{\dfrac{4(\lambda+\mu)^2-\lambda^2}{4(\mu+\lambda)^2}}\begin{bmatrix}1 & \dfrac{\lambda}{2(\mu+\lambda)} & 0\\ \dfrac{\lambda}{2(\mu+\lambda)} & 1 & 0\\ 0 & 0 & \dfrac{2(\mu+\lambda)-\lambda}{4(\mu+\lambda)}\end{bmatrix}$$

$$=\frac{4\mu(2\mu+3\lambda)(\mu+\lambda)}{4(\lambda+\mu)^2-\lambda^2}\frac{1}{2(\mu+\lambda)}\begin{bmatrix}(\mu+\lambda) & \lambda & 0\\ \lambda & (\mu+\lambda) & 0\\ 0 & 0 & \dfrac{2(\mu+\lambda)-\lambda}{2}\end{bmatrix}$$

$$=\frac{2\mu(2\mu+3\lambda)}{4(\lambda+\mu)^2-\lambda^2}\begin{bmatrix}(\mu+\lambda) & \lambda & 0\\ \lambda & (\mu+\lambda) & 0\\ 0 & 0 & \mu+\dfrac{\lambda}{2}\end{bmatrix}$$

Taking into account the plane strain assumptions, the elastic matrix is

$$\boldsymbol{E}=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix}1 & \dfrac{\nu}{1-\nu} & 0\\ \dfrac{\nu}{1-\nu} & 1 & 0\\ 0 & 0 & \dfrac{1-2\nu}{2(1-\nu)}\end{bmatrix}$$

Substituting and separating terms for simplify

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}=\frac{\dfrac{\mu(2\mu+3\lambda)}{\mu+\lambda}\left(1-\dfrac{\lambda}{2(\mu+\lambda)}\right)}{\left(1+\dfrac{\lambda}{2(\mu+\lambda)}\right)\left(1-2\dfrac{\lambda}{2(\mu+\lambda)}\right)}=$$

$$=\frac{\dfrac{\mu(2\mu+3\lambda)}{(\mu+\lambda)}\left(\dfrac{2\mu+2\lambda-\lambda}{2(\mu+\lambda)}\right)}{\left(\dfrac{2\mu+2\lambda+\lambda}{2(\mu+\lambda)}\right)\left(\dfrac{\mu+\lambda-\lambda}{(\mu+\lambda)}\right)}=\frac{\mu(2\mu+3\lambda)(2\mu+\lambda)2(\mu+\lambda)^2}{(2\mu+3\lambda)\mu 2(\mu+\lambda)^2}=2\mu+\lambda$$

$$\frac{\nu}{1-\nu}=\frac{\frac{\lambda}{2(\mu+\lambda)}}{1-\frac{\lambda}{2(\mu+\lambda)}}=\frac{\frac{\lambda}{2(\mu+\lambda)}}{\frac{2\mu+2\lambda-\lambda}{2(\mu+\lambda)}}=\frac{\lambda}{2\mu+\lambda}$$

$$\frac{1-2\nu}{2(1-\nu)}=\frac{1-2\frac{\lambda}{2(\mu+\lambda)}}{2\left(1-\frac{\lambda}{2(\mu+\lambda)}\right)}=\frac{\frac{\mu+\lambda-\lambda}{(\mu+\lambda)}}{2\left(\frac{2\mu+2\lambda-\lambda}{2(\mu+\lambda)}\right)}=\frac{\mu(\mu+\lambda)}{(\mu+\lambda)(2\mu+\lambda)}=\frac{\mu}{2\mu+\lambda}$$

Replacing

$$\boldsymbol{E}=2\mu+\lambda\begin{bmatrix}1 & \frac{\lambda}{2\mu+\lambda} & 0\\ \frac{\lambda}{2\mu+\lambda} & 1 & 0\\ 0 & 0 & \frac{\mu}{2\mu+\lambda}\end{bmatrix}=\begin{bmatrix}2\mu+\lambda & \lambda & 0\\ \lambda & 2\mu+\lambda & 0\\ 0 & 0 & \mu\end{bmatrix}$$

3. Splitting the previous matrix

$$\boldsymbol{E}_{\boldsymbol{\lambda}}=\lambda\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix};\boldsymbol{E}_{\boldsymbol{\mu}}=\mu\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

4. Substituting

$$\boldsymbol{E}_{\boldsymbol{\lambda}}=\frac{E\nu}{(1+\nu)(1-2\nu)}\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\boldsymbol{E}_{\boldsymbol{\mu}}=\frac{E}{2(1+\nu)}\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

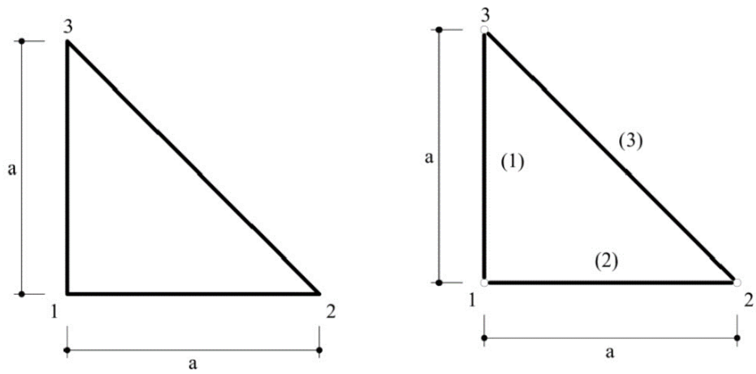
Assignment 3.2:

On “The 3-node Plane Stress Triangle”:

Consider a plane triangular domain of thickness h, with horizontal and vertical edges of length a. Let us consider for simplicity a = 1, h = 1. The material parameters are E, ν. Initially ν is set to zero. Two discrete structural models are considered as depicted in the figure:

a) A plane linear Turner triangle with the same dimensions.

b) A set of three bar elements placed over the edges of the triangular domain. The cross sections for the bars are A1 = A2 and A3.



1. Calculate the stiffness matrices Ktri and Kbar for both discrete models.
2. Is there any set of values for the cross sections A1=A2 and A3 to make both stiffness matrix equivalent: Kbar = Ktri? If not, which are the values that make them more similar?
3. Why these two stiffness matrices are not equal?. Find a physical explanation.
4. Consider nowidering v≠ 0 and extract some conclusions.

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1. For the bar elements, the stiffness matrix in local coordinates

$$\overline{\boldsymbol{K}} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The displacement transformation matrix

$$\boldsymbol{T}^e = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \text{ where } c = cos\varphi; s = sin\varphi$$

Globalization of the element stiffness matrix

$$\boldsymbol{K}^e = (\boldsymbol{T}^e)^T \overline{\boldsymbol{K}}^e \boldsymbol{T}^e$$

$$\boldsymbol{K}^e = \frac{E^e A^e}{L^e}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

For bar (1)

$$c = 0, s = 1, L = a, A = A_1$$

$$\boldsymbol{K}^1 = \frac{EA_1}{a}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

For bar (2)

$$c = 1, s = 0, L = a, A = A_1$$

$$\boldsymbol{K}^2 = \frac{EA_1}{a}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For bar (3)

$$c = -\frac{1}{\sqrt{2}}, s = \frac{1}{\sqrt{2}}, L = \sqrt{a^2 + a^2} = \sqrt{2}a, A = A_3$$

$$\boldsymbol{K}^3 = \frac{EA_3}{\sqrt{2}a}\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

Expanding, for the complete displacement vector

$$\boldsymbol{u} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

$$\boldsymbol{K}^1 = \frac{EA_1}{a}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{K}^2 = \frac{EA_1}{a}\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{K}^3 = \frac{EA_3}{\sqrt{2}a}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5 & -0.5 & 0.5 \\ 0 & 0 & -0.5 & 0.5 & 0.5 & -0.5 \\ 0 & 0 & -0.5 & 0.5 & 0.5 & -0.5 \\ 0 & 0 & 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

Forming the Master Stiffness Equations

$$
\boldsymbol{K_{bar}} = E\begin{bmatrix} \frac{A_1}{a} & 0 & -\frac{A_1}{a} & 0 & 0 & 0 \\ 0 & \frac{A_1}{a} & 0 & 0 & 0 & -\frac{A_1}{a} \\ -\frac{A_1}{a} & 0 & -\frac{A_1}{a}+\frac{A_3}{2\sqrt{2}a} & -\frac{A_3}{2\sqrt{2}a} & -\frac{A_3}{2\sqrt{2}a} & \frac{A_3}{2\sqrt{2}a} \\ 0 & 0 & -\frac{A_3}{2\sqrt{2}a} & \frac{A_3}{2\sqrt{2}a} & \frac{A_3}{2\sqrt{2}a} & -\frac{A_3}{2\sqrt{2}a} \\ 0 & 0 & -\frac{A_3}{2\sqrt{2}a} & \frac{A_3}{2\sqrt{2}a} & \frac{A_3}{2\sqrt{2}a} & -\frac{A_3}{2\sqrt{2}a} \\ 0 & -\frac{A_1}{a} & \frac{A_3}{2\sqrt{2}a} & -\frac{A_3}{2\sqrt{2}a} & -\frac{A_3}{2\sqrt{2}a} & \frac{A_1}{a}+\frac{A_3}{2\sqrt{2}a} \end{bmatrix}
$$

With $\nu = 0$, the Elasticity matrix is

$$
\boldsymbol{E} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = E\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}
$$

The nodal coordinates vector is

$$
\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a \\ 0 \\ 0 \\ a \end{bmatrix}
$$

With

$$
x_{jk} = x_j - x_k \; and \; y_{jk} = y_j - y_k
$$

The element stiffness matrix is

$$
\boldsymbol{K_{tri}} = \frac{h}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix}\boldsymbol{E}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} =
$$

$$
= \frac{h}{2a^2}\begin{bmatrix} -a & 0 & -a \\ 0 & -a & -a \\ a & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & a \\ 0 & a & 0 \end{bmatrix} E\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -a & 0 & a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & a \\ -a & -a & 0 & a & a & 0 \end{bmatrix} =
$$

$$
= \frac{Eh}{2a^2}\begin{bmatrix} \frac{3}{2}a^2 & \frac{1}{2}a^2 & -a^2 & -\frac{1}{2}a^2 & -\frac{1}{2}a^2 & 0 \\ \frac{1}{2}a^2 & \frac{3}{2}a^2 & 0 & -\frac{1}{2}a^2 & -\frac{1}{2}a^2 & -a^2 \\ -a^2 & 0 & a^2 & 0 & 0 & 0 \\ -\frac{1}{2}a^2 & -\frac{1}{2}a^2 & 0 & \frac{1}{2}a^2 & \frac{1}{2}a^2 & 0 \\ -\frac{1}{2}a^2 & -\frac{1}{2}a^2 & 0 & \frac{1}{2}a^2 & \frac{1}{2}a^2 & 0 \\ 0 & -a^2 & 0 & 0 & 0 & a^2 \end{bmatrix} =
$$

$$
= \frac{Eh}{2}\begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{3}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

2. Comparing both matrix

$$
\boldsymbol{K_{bar}} = E\begin{bmatrix} \frac{A_1}{a} & 0 & -\frac{A_1}{a} & 0 & 0 & 0 \\ 0 & \frac{A_1}{a} & 0 & 0 & 0 & -\frac{A_1}{a} \\ -\frac{A_1}{a} & 0 & -\frac{A_1}{a} + \frac{A_3}{2\sqrt{2}a} & -\frac{A_3}{2\sqrt{2}a} & -\frac{A_3}{2\sqrt{2}a} & \frac{A_3}{2\sqrt{2}a} \\ 0 & 0 & -\frac{A_3}{2\sqrt{2}a} & \frac{A_3}{2\sqrt{2}a} & \frac{A_3}{2\sqrt{2}a} & -\frac{A_3}{2\sqrt{2}a} \\ 0 & 0 & -\frac{A_3}{2\sqrt{2}a} & \frac{A_3}{2\sqrt{2}a} & \frac{A_3}{2\sqrt{2}a} & -\frac{A_3}{2\sqrt{2}a} \\ 0 & -\frac{A_1}{a} & \frac{A_3}{2\sqrt{2}a} & -\frac{A_3}{2\sqrt{2}a} & -\frac{A_3}{2\sqrt{2}a} & \frac{A_1}{a} + \frac{A_3}{2\sqrt{2}a} \end{bmatrix}
$$

$$
\boldsymbol{K_{tri}} = \frac{Eh}{2}\begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{3}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

There are many zero terms in the $\boldsymbol{K_{bar}}$ matrix which correlation value in the $\boldsymbol{K_{tri}}$ aren't zero, so there aren't set of values of A_1 and A_2 to make both equal.

Approximating the first 3 diagonal values

$$\frac{A_1}{a} = \frac{3h}{4} \quad \rightarrow A_1 = \frac{3a}{4}h$$

$$-\frac{A_1}{a} + \frac{A_3}{2\sqrt{2}a} = \frac{h}{2} \rightarrow -\frac{\frac{3ah}{4}}{a} + \frac{A_3}{2\sqrt{2}a} = -\frac{3h}{4} + \frac{A_3}{2\sqrt{2}a} = \frac{h}{2}$$

$$\frac{A_3}{2\sqrt{2}a} = \frac{5}{4}h \rightarrow A_3 = \frac{5\sqrt{2}a}{2}h$$

Replacing in $\boldsymbol{K_{bar}}$

$$\boldsymbol{K_{bar}} = \frac{Eh}{2}\begin{bmatrix} \frac{3}{2} & 0 & -\frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 & 0 & -\frac{3}{2} \\ -\frac{3}{2} & 0 & 1 & -\frac{5}{2} & -\frac{5}{2} & \frac{5}{2} \\ 0 & 0 & -\frac{5}{2} & \frac{5}{2} & \frac{5}{2} & -\frac{5}{2} \\ 0 & 0 & -\frac{5}{2} & \frac{5}{2} & \frac{5}{2} & -\frac{5}{2} \\ 0 & -\frac{3}{2} & \frac{5}{2} & -\frac{5}{2} & -\frac{5}{2} & 4 \end{bmatrix}$$

With a virtual displacement $\boldsymbol{u}$ imposed,

$$\boldsymbol{u_v} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} \quad \rightarrow \boldsymbol{K_{tri}u_v} = \boldsymbol{f_{tri}}\,; \boldsymbol{K_{bar}u_v} = \boldsymbol{f_{bar}} = \boldsymbol{f_{tri}} + \boldsymbol{e_{rr}}$$

$$\boldsymbol{e_{rr}} = \boldsymbol{f_{bar}} - \boldsymbol{f_{tri}} = \boldsymbol{K_{bar}u_v} - \boldsymbol{K_{tri}u_v} = (\boldsymbol{K_{bar}} - \boldsymbol{K_{tri}})\boldsymbol{u_v}$$

$$\boldsymbol{e_{rr}} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{5}{2} & \frac{5}{2} & -\frac{5}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} & -2 & -2 & \frac{5}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} & -2 & -2 & \frac{5}{2} \\ 0 & \frac{5}{2} & -\frac{5}{2} & \frac{5}{2} & \frac{5}{2} & -3 \end{bmatrix}\boldsymbol{u_v}$$

As the most values are different of zero, all the components of the force vector are going to have an error, depending of the displacement vector applied. Probably you can adjust the values of A_1 and A_3 to get a minor error.

3. The matrixes are not equal because the discrete bar model doesn't take into account the effect of the displacement of the nodes inside the area of the triangle. The linear Turner triangle take this into account with the shape functions. Also, bar elements are able to support axial forces only but triangular elements are able to support shear and axial forces.

4. With $\nu \neq 0$, the Elasticity matrix is

$$\boldsymbol{E} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

The element stiffness matrix is

$$\boldsymbol{K_{tri}} = \frac{h}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \boldsymbol{E} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} =$$

$$= \frac{h}{2a^2}\begin{bmatrix} -a & 0 & -a \\ 0 & -a & -a \\ a & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & a \\ 0 & a & 0 \end{bmatrix} \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} -a & 0 & a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & a \\ -a & -a & 0 & a & a & 0 \end{bmatrix} =$$

$$= \frac{Eh}{2}\begin{bmatrix} \frac{1}{2}\frac{-3+\nu}{\nu^2-1} & -\frac{1}{2(-1+\nu)} & \frac{1}{\nu^2-1} & -\frac{1}{2(\nu+1)} & -\frac{1}{2(\nu+1)} & \frac{\nu}{\nu^2-1} \\ -\frac{1}{2(-1+\nu)} & \frac{1}{2}\frac{-3+\nu}{\nu^2-1} & \frac{\nu}{\nu^2-1} & -\frac{1}{2(\nu+1)} & -\frac{1}{2(\nu+1)} & \frac{1}{\nu^2-1} \\ \frac{1}{\nu^2-1} & \frac{\nu}{\nu^2-1} & -\frac{1}{\nu^2-1} & 0 & 0 & -\frac{\nu}{\nu^2-1} \\ -\frac{1}{2(\nu+1)} & -\frac{1}{2(\nu+1)} & 0 & \frac{1}{2(\nu+1)} & \frac{1}{2(\nu+1)} & 0 \\ -\frac{1}{2(\nu+1)} & -\frac{1}{2(\nu+1)} & 0 & \frac{1}{2(\nu+1)} & \frac{1}{2(\nu+1)} & 0 \\ \frac{\nu}{\nu^2-1} & \frac{1}{\nu^2-1} & -\frac{\nu}{\nu^2-1} & 0 & 0 & -\frac{1}{\nu^2-1} \end{bmatrix}$$

Comparing with de previous matrix, the main changes appears in the 3rd and 6th rows and column of the matrix. In the matrix with $\nu = 0$, the nodal forces horizontal in node 2 and vertical in node 3 depends on the deformation in bar 2 and the deformation in bar 1, respectively. Instead, in the matrix with $\nu \neq 0$, appears more terms, due to the Poisson effect in bar 3.

$$\boldsymbol{K_{tri}} = \frac{Eh}{2}\begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{3}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$