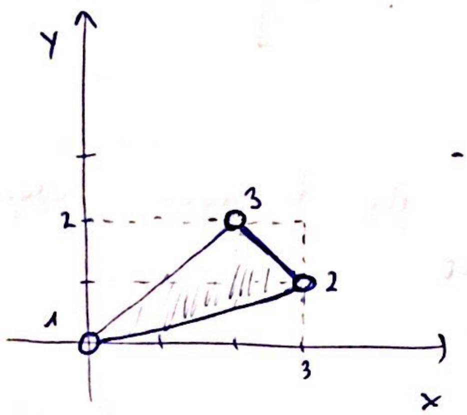
- Oscar Salvador -

Assignment 3

3.1



- plane stress triangle -

1 (0, 0)
2 (3, 1)
3 (2, 2)

$$A = 3\cdot 2 - \frac{3\cdot 1}{2} - \frac{1\cdot 1}{2} - \frac{2\cdot 2}{2} = 2$$

$$E = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}, \quad h = 1 \text{ (thickness)}$$

K_e computation

$$K^e = \int_{\Omega^e} h\, B^T E B \, d\Omega^e$$

B (strain-displacement matrix)

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}$$

shape functions (linear triangle)

$N_i = \xi_i \quad i = 1, 2, 3$ (triangular coordinates)

$$B = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$* \; x_{ij} = x_i - x_j$

$$B = \frac{1}{2A}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix}$$

As B, E are constant along the triangular element:

$$K^e = B^T \cdot E \cdot B \cdot \cancel{A} \int_{\Omega^e} \overset{\text{constant}}{h}\, d\Omega^e$$

$$K^e = \frac{h}{4A}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \\ -1 & 0 & 3 \\ 0 & 3 & -1 \end{bmatrix}\begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & 1 \end{bmatrix}$$

$$= \frac{h}{4A}\begin{bmatrix} -100 & -25 & -50 \\ -25 & -100 & -50 \\ 200 & 50 & -100 \\ -50 & -200 & 100 \\ -100 & -25 & 150 \\ 75 & 300 & -50 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 18.75 & 9.375 & -12.50 & -6.25 & -6.25 & -3.125 \\ 9.375 & 18.75 & 6.25 & 12.50 & -15.625 & -31.25 \\ -12.50 & 6.25 & 75.00 & -37.50 & -62.50 & 31.25 \\ -6.25 & 12.50 & -37.50 & 75.00 & 43.75 & -87.50 \\ -6.25 & -15.625 & -62.50 & 43.75 & 68.75 & -28.125 \\ -3.125 & -31.25 & 31.25 & -87.50 & -28.125 & 118.75 \end{bmatrix}$$

2. Sum of rows and columns

Rows 1, 3, 5

$$18.75 + 9.375 + (-12.50) + (-6,25) + -(6.25) + (-3.125)$$
$$+ (-12.50) + 6.25 + 75.00 + (-37.50) + (-62.50) + 31.25$$
$$+ (-6.25) + (-15.625) + (-62.50) + 43.75 + 68.75 + (-28.125)$$
$$= 0$$

Columns 1, 3, 5

$$\begin{bmatrix} 18.75 \\ 9.375 \\ -12.50 \\ -6.25 \\ -6.25 \\ -3.125 \end{bmatrix} + \begin{bmatrix} -12.50 \\ 6.25 \\ 75.00 \\ -37.50 \\ -62.50 \\ 31.25 \end{bmatrix} + \begin{bmatrix} -6.25 \\ -15.625 \\ -62.50 \\ 43.75 \\ 68.75 \\ -28.125 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

the same for rows and columns 2, 4, 6

Discussion

This system is not constrained. For that reason, any rigid body motion is possible without any ~~external~~ nodal force.

For $Ku = f$ being $u = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \rightarrow f = \{0\}$

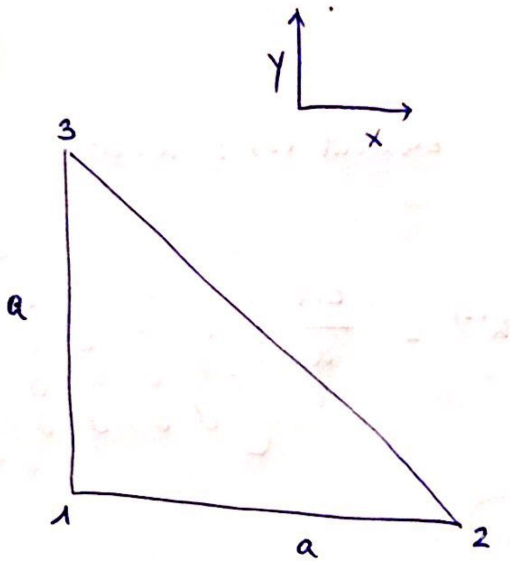
~~1, 3, 5 rows represents the x-~~ ~~displacements~~ ~~forces, as well as 2, 4, 6 represent the y-~~ ~~displacements~~ ~~forces of the three nodes.~~

~~1, 3, 5 columns~~

1, 3, 5 columns, rows $\rightarrow$ x-axis

2, 4, 6 columns, rows $\rightarrow$ y-axis

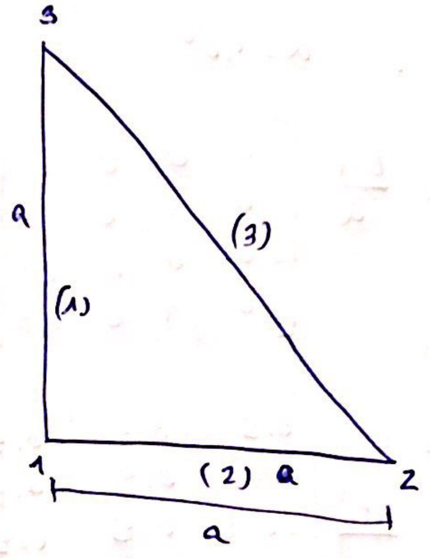
3.2



- Turner triangle -

$a = h = 1$

$E, \nu \rightarrow$ Initially $\nu = 0$



- Three bars -

$\underbrace{A_1 = A_2}_{A}, \ \underbrace{A_3}_{A'}$

a) stiffness matrices calculation

$\boxed{K_{triangle}}$

$$k^e = \frac{h}{4A} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} [E] \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

* $h = 1$

$A = \frac{a^2}{2} = 1/2$

$$[E] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

(plane stress case)

if $\nu = 0$

$$= \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E/2 \end{bmatrix}$$

$y_{23} = 0 - a$

$x_{32} = -a$

$y_{31} = a$

$x_{13} = 0$

$y_{12} = 0$

$x_{21} = a$

$$K = \frac{E}{} \begin{bmatrix} 0.75 & 0.25 & -0.50 & -0.25 & -0.25 & 0 \\ 0.25 & 0.75 & 0 & -0.25 & -0.25 & -0.5 \\ -0.50 & 0 & 0.50 & 0 & 0 & 0 \\ -0.25 & -0.25 & 0 & 0.25 & 0.25 & 0 \\ -0.25 & -0.25 & 0 & 0.25 & 0.25 & 0 \\ 0 & -0.50 & 0 & 0 & 0 & 0.5 \end{bmatrix}$$

$\boxed{K_{bar}}$

Element 1 : $\alpha = 90°$

$$k^{(1)} = \frac{EA}{a}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Element (2) : $\alpha = 0°$

$$k^{(2)} = \frac{EA}{a}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 3 :

$$\frac{EA'}{\sqrt{2}a}\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

Global K $(a=1)$

$$K = E\begin{bmatrix} \cancel{E}A & 0 & -\cancel{E}A & 0 & 0 & 0 \\ & \cancel{E}A & 0 & 0 & 0 & -\cancel{E}A \\ & & A + \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} \\ & & & \frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} \\ & \text{symm} & & & \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} \\ & & & & & A + \frac{A'}{2\sqrt{2}} \end{bmatrix}$$

b) Similarity

It isn't exist any value ~~that~~ of A, A' that makes $K_{triang.} = K_{bar}$.

If we use $\left.\begin{matrix} A_1 = A_2 = A = 0.25 \\ \text{and} \\ A' = \sqrt{2}/2 \end{matrix}\right\}$ $\underbrace{k_{33},\ k_{44},\ k_{55},\ k_{66}}$ these values of the diagonal coincide.

c) Why K_{bar} / K_{triang} are not equivalent?

these two stiffness matrices cannot be equivalent because their respective mathematical models are completely different.

The 'plane stress model' have stiffness in more directions than the 'bars model' as its constitutive equation shows.

- plane stress -

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix}$$

- bar model -

$$\sigma = E\varepsilon \quad (1D)$$

d) $\nu \neq 0$

K_{bar} is the same.

$\boxed{K_{triangle}}$

$$K = \frac{1}{2} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$k = \frac{E}{2(1-\nu^2)} \begin{bmatrix} (3/2 - \nu/2) & (\nu/2 + 1/2) & -1 & (\nu/2 - 1/2) & (\nu/2 - 1/2) & -\nu \\ & (3/2 - \nu/2) & -\nu & (\nu/2 - 1/2) & (\nu/2 - 1/2) & -1 \\ & & 1 & 0 & 0 & \nu \\ & \text{symm} & & (1/2 - \nu/2) & (1/2 - \nu/2) & 0 \\ & & & & (1/2 - \nu/2) & 0 \\ & & & & & 1 \end{bmatrix}$$

Discussion

We can observe that the stiffness matrix is completely different considering Poison's ratio ($\nu \neq 0$). This ratio relates the deformation between two different directions. Therefore, this ratio adds a sort of 'transversal stiffness' that makes the 'plane stress model' more different than the 'truss model'.