CSMD: Assignment 3

Juan Pedro Roldán

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1 Plane stress problem

1.1 Inverse relation for E and ν

Considering the expressions of the Lamé constants in terms of λ and ν, we can use the relation for μ $E = \mu\ 2(1+\nu)$ into λ

$$\lambda = \frac{\mu\ 2(1+\nu)\ \nu}{(1+\nu)(1-2\nu)}$$

And then we isolate ν:

$$\frac{\lambda}{2\ \mu} = \frac{\nu}{1-2\nu}$$
$$\frac{\lambda - 2\lambda\nu}{2\ \mu} = \nu$$
$$\frac{\lambda}{2\ \mu} = \nu + \frac{\lambda\nu}{\mu} = \nu(\frac{\lambda+\mu}{\mu})$$
$$\frac{\lambda}{2\ (\lambda+\mu)} = \nu$$

And we can substitute ν into the expression of μ to obtain E:

$$E = 2\mu\ (1 + \frac{\lambda}{2\ (\lambda+\mu)}) = 2\mu\frac{2\lambda+2\mu+\lambda}{2\ (\lambda+\mu)} = \mu\ \frac{3\lambda+2\mu}{\lambda+\mu}$$

1.2 Elastic matrix for plane stress and plane strain

Substituting the previously derived relations into the plane stress elastic matrix we can express it in terms of μ and λ:

$$
\begin{aligned}
&\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix} = \\
=&\mu\ \frac{3\lambda+2\mu}{\lambda+\mu}\frac{4(\lambda+\mu)^2}{4(\lambda+\mu)^2-\lambda^2}\begin{bmatrix}1 & \frac{\lambda}{2\ (\lambda+\mu)} & 0\\ \frac{\lambda}{2\ (\lambda+\mu)} & 1 & 0\\ 0 & 0 & \frac{2-\lambda}{4(\lambda+\mu)}\end{bmatrix} = \\
=&4\mu\frac{3\lambda+2\mu}{3\lambda^2+8\lambda\mu+4\mu^2}\begin{bmatrix}1 & \frac{\lambda}{2} & 0\\ \frac{\lambda}{2} & 1 & 0\\ 0 & 0 & \frac{2-\lambda}{4}\end{bmatrix}
\end{aligned}
$$

And doing the same for the plane strain elastic matrix

$$
\begin{aligned}
&\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix}1 & \frac{\nu}{1-\nu} & 0\\ \frac{\nu}{1-\nu} & 1 & 0\\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)}\end{bmatrix} = \\
=&\lambda+2\mu\begin{bmatrix}1 & \frac{\lambda}{\lambda+2\mu} & 0\\ \frac{\lambda}{\lambda+2\mu} & 1 & 0\\ 0 & 0 & \frac{\mu}{2(\lambda+2\mu)}\end{bmatrix} = \begin{bmatrix}\lambda+2\mu & \lambda & 0\\ \lambda & \lambda+2\mu & 0\\ 0 & 0 & \frac{\mu}{2}\end{bmatrix}
\end{aligned}
$$

1.3 Plane strain elastic matrix splitting

This last matrix can be split into two different matrices E_λ and E_μ

$$
\mathbf{E}_\lambda + \mathbf{E}_\mu = \begin{bmatrix}\lambda & \lambda & 0\\ \lambda & \lambda & 0\\ 0 & 0 & 0\end{bmatrix} + \begin{bmatrix}2\mu & 0 & 0\\ 0 & 2\mu & 0\\ 0 & 0 & \frac{\mu}{2}\end{bmatrix}
$$

1.4 Split matrices in terms of E and ν

$$
\mathbf{E}_\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}
$$

$$
\mathbf{E}_\mu = \frac{E}{4(1+\nu)}\begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{bmatrix}
$$

2 Three node triangle

2.1 Stiffness matrices

The general expression of the Turner triangle stiffness matrix is:

$$\mathbf{K}_t^e = \frac{h}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

Computing the values of $\mathrm{y_{ij}}$ and $\mathrm{x_{ij}}$ from the triangle data and substituting the expression of E for the plane stress case we obtain:

$$\mathbf{K}_t^e = \frac{1}{2}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & \frac{E}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

The final value of $\mathbf{K}^{\mathrm{e}}{}_{\mathrm{t}}$ is

$$\mathbf{K}_t^e = \frac{E}{2}\begin{bmatrix} 1.5 & 0.5 & -1 & -0.5 & -0.5 & 0 \\ 0.5 & 1.5 & 0 & -0.5 & -0.5 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

If we consider a triangle formed by three pin-jointed bars, we can express the stiffness matrix of the system (assembling accordingly the elemental stiffness matrices, as done in assignment 1):

$$\mathbf{K}_b^e = \frac{E}{2}\begin{bmatrix} 2A_1 & 0 & 0 & 0 & -2A_1 & 0 \\ 0 & 2A_1 & 0 & -2A_1 & 0 & 0 \\ 0 & 0 & A_3 & -A_3 & -A_3 & A_3 \\ 0 & -2A_1 & -A_3 & 2A_1 + A_3 & A_3 & -A_3 \\ -2A_1 & 0 & -A_3 & A_3 & A_3 + 2A_1 & -A_3 \\ 0 & 0 & A_3 & -A_3 & -A_3 & A_3 \end{bmatrix}$$

2.2 Stiffness matrices comparison

We can directly see that it is impossible to make both matrices ($\mathbf{K}_t^e$ and $\mathbf{E}_b^e$) equal, as some nonzero components of $\mathbf{E}_t^e$ are zero in $\mathbf{E}_b^e$. Possible values to make them similar are $A_1 = 0.75$ and $A_3 = 1$. This way, most of the diagonal components are the same as in $\mathbf{E}_t^e$. We would have the next matrices:

$$\mathbf{K}_t^e = \frac{E}{2}\begin{bmatrix} 1.5 & 0.5 & -1 & -0.5 & -0.5 & 0 \\ 0.5 & 1.5 & 0 & -0.5 & -0.5 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{K}_b^e = \frac{E}{2}\begin{bmatrix} 1.5 & 0 & 0 & 0 & -1.5 & 0 \\ 0 & 1.5 & 0 & -1.5 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & -1.5 & -1 & 2.5 & 1 & -1 \\ -1.5 & 0 & -1 & 1 & 2.5 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

2.3 Stiffness matrices comparison analysis

These two matrices cannot be equal because they cannot represent the same type of physical element (a plate, in this case). When plates are discretized by using linear bars, we are losing the correlation between forces and displacements that appear in the formulation of the solid plate stiffness matrix. And it makes sense, as the plate spans the whole space between the edges of the triangle, while the bar has (literally) a hole in between. Looking again at the stiffness matrices, we can see that, in general, it does not take into account the displacements in those directions perpendicular to the bar direction. For instance, the first row of $\mathbf{K}_b^e$ only takes into account the horizontal displacements of nodes 1 and 3 for external horizontal force on node 1.

2.4 Widening

The triangle stiffness matrix with widening effect can be computed as

$$\mathbf{K}_t^e = \frac{1}{2}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix} =$$

$$\mathbf{K}_t^e = \frac{E}{2(1-\nu^2)}\begin{bmatrix} \frac{3+\nu}{2} & \frac{\nu+1}{2} & -1 & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -\nu \\ \frac{1+\nu}{2} & \frac{3+\nu}{2} & -\nu & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -1 \\ -1 & -\nu & 1 & 0 & 0 & \nu \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ -\nu & -1 & \nu & 0 & 0 & 1 \end{bmatrix}$$

The new stiffness matrix increases stiffness, as we can see by the fact that the constant term $\frac{E}{2(1-\nu^2)}$ increases as ν increases. With respect to the stiffness matrix components, there are more non-zero components, and there is also some balancing between terms that increase their value (such as K_{11} and those that decrease it (K_{14}, for instance).